Modeling Hierarchical Spatial Interdependence for Limited Dependent Variables*

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Abstract

Scholars are often interested in questions that examine interactions between units at two different levels. For example, counties are nested within states and diffusion processes might take place at both levels of analysis. Building on recent research from the spatial econometrics and multilevel modeling literature, we propose a method for modeling spatial interdependencies in two hierarchical levels with binary and ordered outcomes. We propose a Bayesian approach that estimates spatial autoregressive parameters at both hierarchical levels, and provide software to estimate this model. Our Monte Carlo results demonstrate that failing to account for the nested structure of the data leads to biased parameter estimates. We demonstrate the utility of our approach by analyzing the causes of civil rights protests in the United States in the 1960s.

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1 Introduction

Analyzing data that have a nested structure is popular in Political Science. Two examples of data that contain a nested structure include units that are each observed across multiple time periods (e.g., time-series-cross-section data), and units at different levels of analysis, (e.g., voters in a county). Due to the hierarchical nature of the data, estimating models that fail to account for the nested structure of the data can lead to inaccurate inferences.

Another popular methodological approach that has been garnering popularity is the spatial analysis of political data. Many Political Science theories involve the diffusion of some policy across units or an occurrence of a certain phenomenon in some units affecting outcomes in other units. For example, Franzese, Hays and Cook (2016) model the civil war diffusion process in Sub-Saharan Africa and estimate the increased probability of civil war in Guinea-Bissau due to internal instability in Senegal. A budding vein of research now combines advances in spatial econometrics and hierarchical modeling to model diffusion at potentially multiple levels of analysis.

We contribute by introducing a spatial-hierarchical strategy for categorical outcomes that are binary or ordered. We introduce a hierarchical spatial probit approach for binary and ordered outcomes.[1] We use a well-known "data augmentation algorithm" to extend the hierarchical spatial model for continuous outcomes to binary and ordered outcomes (Albert and Chib, 1993). The entire process ultimately involves generating simulated values of the latent outcome variable $\mathbf{y}^*$ by using a multivariate truncated normal distribution, and then sequentially drawing from full conditional posterior distributions for other parameters of

[1] Our proposed methodology is easily extendable to cases with ordered outcomes. We summarize this in the paper and we discuss this in detail in Appendix I.

interest.[2]

We reach two main conclusions from a series of Monte Carlo experiments. First, as expected, estimating a spatial autoregressive model that accounts for diffusion at only one level results in inaccurate inferences when there is diffusion at multiple levels. Second, we find that our proposed hierarchical spatial model that accounts for diffusion across units at multiple levels can be used as a general modeling strategy. We find that estimating our recommended model results in accurate inferences in such a scenario; the spatial parameter for the level in which diffusion does not occur is statistically insignificant. While intuitive, this is important because the lack of false positives highlights how our model can be used as a robustness check to verify the lack of diffusion among higher-level units.[3]

Spatial Econometrics and Hierarchical Models

For the most part, hierarchical models and spatial econometrics have developed as two distinct fields.[4] The hierarchical modeling literature implicitly assumes that units in a common group share some similar characteristics. The typical example often used is that students in the same classroom may often have correlated errors because they have the same teacher and share similar experiences within the classroom. A common approach in economics has been to model the unobserved relationship through the use of robust standard errors (e.g., Angrist, Bettinger and Kremer, 2006). In contrast, hierarchical models explicitly deal with this by nesting lower-level units within high-level units.

[2]We provide an R package that estimates this model and calculates substantial effects of interest.

[3]This is assuming the estimator satisfies all assumptions for unbiased parameter estimates and accurate standard errors.

[4]Similar to Gelman et al. (2013), we use the terms multilevel models and hierarchical models interchangeably.

The focus of the spatial econometrics literature is to explicitly model theoretically interesting diffusion processes. For example, Simmons and Elkins (2004) examines how liberal economic policies have diffused across countries over time. There have been recent advances in the spatial econometrics literature that focus on combining multilevel and spatial modeling. For example, Dong and Harris (2015) proposed a hierarchical spatial autoregressive model with which they estimated the effects of various factors on the leasing price of land parcels in China, where land parcels are grouped into various districts. Advances in spatial and hierarchical modeling combined with data availability has created opportunities to improve our understanding of politics. For example, while Mazumder (2018) investigates the persistent effects of civil rights protests, scholars might also want to investigate the conditions why protests are likely to occur in some counties but not in others. In such a case, scholars might want to consider the following two features. First, counties are nested within states. Different states might have different political, social and economic conditions that would affect the common baseline propensity of civil rights protests to occur in counties within the same state. Secondly, there might *potentially* be a spillover effect between states. In other words, there could potentially be a weak diffusion process between states themselves as well. The model we propose takes into account of the nested structure as well as providing a way to test whether there might be a diffusion process among higher-level units as well.

2 Spatial Autoregressive Model

One of the most popular models used in spatial econometrics is the spatial autoregressive (SAR) model, [5] which can be expressed as

$$\mathbf{y} = \rho\mathbf{W}\mathbf{y} + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \tag{1}$$

where $\mathbf{y}$ is an $N \times 1$ vector of outcomes in all N units, ρ is the spatial autoregressive coefficient, $\mathbf{X}$ is a matrix of predictors with dimensions $N \times K$, $\boldsymbol{\beta}$ is an $K \times 1$ vector, and $\mathbf{W}$ is an $N \times N$ spatial weights matrix.

This model has been used across a broad range of applications that seek to understand how the outcome in one unit affects those in others, such as party positions (Williams and Whitten, 2015) and military spending (George and Sandler, 2018). However, there are cases in which scholars may be interested in modeling the relationship between outcomes of higher-level units in addition to the diffusion in outcomes among lower-level units. This is similar to the motivation behind the use of hierarchical models, but a little different in terms of its focus. For example, hierarchical models might be used to study the effect of voter characteristics on vote choice. Naturally, voters are grouped into different states and the errors amongst the voters in a given state might be correlated. One advantage of hierarchical modeling has been to increase efficiency by partial pooling (Ghitza and Gelman, 2013).[6]

[5] Readers interested in the full range of commonly models used in spatial econometrics can refer to Halleck Vega and Elhorst (2015).

[6] Partial pooling may be thought of as "compromising between the two extremes of excluding a categorical predictor from a model (complete pooling), or estimating separate models within each level of the categorical predictor (no pooling)." (Gelman and Hill, 2006, 252)

3 Hierarchical Spatial Probit Autoregressive Model

We propose a binary outcome model for estimating spatial autoregressive coefficients that accounts for the hierarchical nature of data. Our proposed hierarchical spatial autoregressive (HSAR) probit model is an extension of the work of Dong and Harris (2015), who focused on the continuous outcomes case.[7]

We can first conceptualize the outcome as a continuous latent variable, $\mathbf{y}^*$, such that

$$\mathbf{y}^* = \rho \mathbf{W} \mathbf{y}^* + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\Delta}\boldsymbol{\theta} + \boldsymbol{\epsilon}, \tag{2}$$

where there are $i = 1, ..., N$ lower-level units nested in $j = 1, ..., J$ higher-level units. $\mathbf{y}^*$ is a $N \times 1$ vector of the latent, continuous outcome variable, $\mathbf{W}$ is an $N \times N$ spatial weights matrix for lower-level units, $\mathbf{X}$ is an $N \times K$ matrix of covariates, $\boldsymbol{\beta}$ is a $K \times 1$ vector of parameters, $\boldsymbol{\Delta}$ is a $N \times J$ matrix mapping lower-level units to higher-level units, $\boldsymbol{\theta}$ is a $J \times 1$ vector of random effects, and $\boldsymbol{\epsilon}$ is an $N \times 1$ vector of white noise.[8]

The model assumes that $\boldsymbol{\theta}$ follows its own autoregressive process and that there is a

[7] Without loss of generality, our model is a stylized version in which higher-level predictors have been omitted. As we show later, this is to make our model easily comparable to the original SAR probit model. Practitioners can easily extend the model presented here to include covariates for the higher-level units.

[8] As an example of a $\boldsymbol{\Delta}$ matrix, consider a 6×3 $\boldsymbol{\Delta}$ matrix consisting of 6 lower-level units with 2 lower-level units in each higher-level unit. This can be represented by the following $\boldsymbol{\Delta}$ matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

spatial weights matrix $\mathbf{M}$ of dimensions $J \times J$ as shown below

$$\boldsymbol{\theta} = \lambda \mathbf{M} \boldsymbol{\theta} + \mathbf{u} \Rightarrow \boldsymbol{\theta} = (\mathbf{I} - \lambda \mathbf{M})^{-1} \mathbf{u} \tag{4}$$

$$\mathbf{u} \sim \mathcal{N}(\mathbf{0}, \sigma_u^2 \mathbf{I}_J) \tag{5}$$

$$\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_N) \tag{6}$$

$$\boldsymbol{\theta} \sim \mathcal{N}(0, \sigma_u^2 (\mathbf{B}' \mathbf{B})^{-1}) \text{ where } \mathbf{B} \equiv \mathbf{I}_J - \lambda \mathbf{M} \tag{7}$$

As is standard in the literature, we assume that we observe a value of 1 for the binary outcome, y_{ij}, if the latent variable y_{ij}^* is (weakly) positive and 0 otherwise:

$$y_{ij} = 1 \Longleftrightarrow y_{ij}^* \geq 0 \tag{8}$$

$$y_{ij} = 0 \Longleftrightarrow y_{ij}^* < 0 \tag{9}$$

Similar to Dong and Harris (2015), we estimate the parameters of interest by adopting a Bayesian approach and implementing Markov Chain Monte Carlo (MCMC) methods. In particular, we use the standard Metropolis-within-Gibbs algorithm in which the parameters are estimated using a Gibbs algorithm whenever the full conditional distributions are of a known form and with a Metropolis algorithm when the distribution is not of a recognizable form. This is standard in the Bayesian literature and has also been used in spatial econometrics to estimate the standard SAR model (LeSage and Pace, 2009).[9]

[9]The Metropolis-within-Gibbs sampling technique has been the standard algorithm for estimating spatial econometric models when using Bayesian methods. We are aware that Bayesian statistics has seen a development of various Markov Chain Monte Carlo algorithms. In particular, the Hamiltonian Monte Carlo algorithm popularized with the development of Stan (Carpenter et al., 2017) has seen wide usage for estimating various models due to its flexibility and ease of use. However, we found that Stan is too computationally inefficient for estimating spatial econometric models with a large number of observations. Wolf, Anselin and

Estimating a binary outcome model using probit is a simple extension of estimating a linear model under the Bayesian framework. If we *assume* that we observe y^*, we can proceed as usual in estimating the relevant parameters of interest. This is the insight that Albert and Chib (1993) provide in estimating the probit model and has been referred in the Bayesian literature as data augmentation based on Tanner and Wong (1987). Of course, since we do not actually observe y^*, we need to generate them. Intuitively, the appropriate values of y^* can be generated under the restriction that y_i^* is positive (negative) if the observed y_i is 1 (0). Conceptually, this involves using a truncated normal distribution which can be implemented through other more efficient sampling algorithms such as inversion sampling.[10] It is well-known that the rejection sampling can suffer from computational inefficiencies (Lynch, 2007) and often a form of inversion sampling is implemented in practice.

However, there is a crucial difference when estimating the probit model in the spatial econometrics literature. In contrast to the standard probit model, the errors are no longer assumed to be independent and the error structure is more complex. We need to thus use truncated multivariate normal distribution to generate the latent variables $\mathbf{y}^*$ instead of (independent) truncated univariate normal distributions as is the case with the standard probit model (e.g., Geweke, 1991). Standard results in statistics demonstrate that conditional distributions of a truncated multivariate normal distribution can be reduced to a truncated univariate normal distribution (Geweke, 1991; Kotecha and Djuric, 1999). Thus, a type of Gibbs sampling procedure can be implemented to sample from a truncated multivariate

Arribas-Bel (2018) suggest that this is due to the computational burden of calculating the log determinant term for each leapfrog step. We refer the readers to Wolf, Anselin and Arribas-Bel (2018) for more details on this matter.

[10] An inversion sampling mechanism involving using a draw from a uniform distribution and the inverse-distribution function (Lynch, 2007, 203).

normal distribution.[11]

The likelihood can be represented as follows for the probit model with binary outcomes :(Jackman, 2009)

$$\mathcal{L}(\beta; y, X) = p(y|X, \beta) = \prod_{i=1}^{n} F(x_i\beta)^{y_i}[1 - F(x_i\beta)^{1-y_i}] \tag{10}$$

where $F(\cdot)$ represents the cumulative normal distribution.

Deriving the posterior distribution for the probit model in practice requires the use of the data augmentation approach for probit models (Tanner and Wong, 1987; Albert and Chib, 1993). This essentially involves estimating a new set of parameters, namely the set of N latent variables (e.g., Smith and LeSage, 2004).

We take a similar approach by adapting the derivations from Dong and Harris (2015). Our first assumption is that the variance of the error, σ_e^2, is 1 for identification purposes. Second, we assume that there is no covariance between $\boldsymbol{\epsilon}$ and $\boldsymbol{\theta}$ (the vector of random effects). These can be written as:

$$Var(\boldsymbol{\epsilon}) = \mathbf{I}_N \tag{11}$$

$$Cov(\boldsymbol{\epsilon}, \boldsymbol{\theta}) = 0 \tag{12}$$

Based on these above assumptions, the variance-covariance matrix of $\mathbf{y}^*$ in Dong and Harris (2015):

[11] In practice, this can be implemented conveniently using the package **tmvtnorm** in **R** (Wilhelm and Manjunath, 2010).

$$Var(\mathbf{y}^*) = (\mathbf{I}_N - \rho\mathbf{W})^{-1}Var(\boldsymbol{\Delta\theta} + \boldsymbol{\epsilon})((\mathbf{I}_N - \rho\mathbf{W})^{-1})' \tag{13}$$

$$= \mathbf{A}^{-1}Var(\boldsymbol{\Delta\theta} + \boldsymbol{\epsilon})(\mathbf{A}^{-1})', \text{ where } \mathbf{A} \equiv \mathbf{I}_N - \rho\mathbf{W} \tag{14}$$

$$= \mathbf{A}^{-1}\Big(\boldsymbol{\Delta}Var(\boldsymbol{\theta})\boldsymbol{\Delta}' + Var(\boldsymbol{\epsilon}) + 2Cov(\boldsymbol{\Delta\theta}, \boldsymbol{\epsilon})\Big)(\mathbf{A}^{-1})' \tag{15}$$

$$= \mathbf{A}^{-1}\Big(\boldsymbol{\Delta}Var(\boldsymbol{\theta})\boldsymbol{\Delta}' + Var(\boldsymbol{\epsilon})\Big)(\mathbf{A}^{-1})' \quad \because \quad \text{Equation 12} \tag{16}$$

$$= \mathbf{A}^{-1}\Big(\boldsymbol{\Delta}(\mathbf{B}'\mathbf{B})^{-1}\boldsymbol{\Delta}' + \mathbf{I}_N\Big)(\mathbf{A}^{-1})' \equiv \mathbf{V} \tag{17}$$

where, $\mathbf{B} \equiv \mathbf{I}_J - \lambda\mathbf{M}$ as per equation 7. As will be seen later, $\mathbf{V}$ will play an important role in using the truncated multivariate normal distributions for generating $\mathbf{y}^*$.[12]

4 Estimating the Model

We adopt a Bayesian Markov Chain Monte Carlo (MCMC) method for model estimation. The basic Bayesian identity used for estimating unknown parameters is (Gelman et al., 2013):

$$p(\boldsymbol{\Theta}|\mathbf{Y}) \propto \mathcal{L}(\mathbf{Y}|\boldsymbol{\Theta}) \times \pi(\boldsymbol{\Theta}) \tag{18}$$

$$\text{posterior} \propto \text{likelihood} \times \text{prior} \tag{19}$$

[12] In equations 13 and 15, we use the rule $Cov(\mathbf{Ax}) = \mathbf{AxA}'$.

We employ diffuse priors to allow the likelihood to be dominant in estimating the parameters.[13] The priors may be specified as follows:

$$\pi(\boldsymbol{\beta}) \sim \mathcal{N}(\mathbf{c}_0, \mathbf{T}_0) \tag{20}$$

$$\pi(\rho) \sim \mathcal{U}\left[\frac{1}{\nu_{\min}}, 1\right] \propto 1 \tag{21}$$

$$\pi(\lambda) \sim \mathcal{U}\left[\frac{1}{\nu^*_{\min}}, 1\right] \propto 1 \tag{22}$$

$$\pi(\boldsymbol{\theta}|\lambda) \sim \mathcal{N}\Big(\mathbf{0}, ((\mathbf{I} - \lambda\mathbf{M})'(\mathbf{I} - \lambda\mathbf{M}))^{-1}\Big) \tag{23}$$

where $\mathbf{A} = \mathbf{I}_N - \rho\mathbf{W}$, $\nu_{\min}$ is the minimum eigenvalue of the spatial weights matrix for the lower-level units, $\mathbf{W}$, and $\nu^*_{\min}$ is the minimum eigenvalue of the spatial weights matrix of high-level units, $\mathbf{M}$. We employ diffuse priors to let the data dominate the posterior. This approach is standard and has been adopted by spatial econometrics researchers in past works (LeSage and Pace, 2009). In practice, this involves centering the prior for $\boldsymbol{\beta}$ on a vector of zero's for the mean $(\mathbf{c}_0)$ with a large variance $(\mathbf{T}_0)$.

The likelihood function in terms of the latent variable $\mathbf{y}^*$ may be specified as follows:

$$\mathcal{L}(\mathbf{y}^*|\rho, \lambda, \boldsymbol{\beta}, \boldsymbol{\theta}, \sigma_u^2) = (2\pi)^{-N/2}|\mathbf{A}| \exp\left\{-\frac{1}{2}(\mathbf{A}\mathbf{y}^* - \mathbf{X}\boldsymbol{\beta} - \boldsymbol{\Delta\theta})'(\mathbf{A}\mathbf{y}^* - \mathbf{X}\boldsymbol{\beta} - \boldsymbol{\Delta\theta})\right\} \tag{24}$$

The respective conditional distributions for each parameter or set of parameters based

[13] The basic derivations for the continuous outcome are detailed in Dong and Harris (2015). We reproduce the derivations here with some minor changes such as working with the latent variable y^* instead of y and constraining σ_e^2 to 1 necessary for identification purposes.

on the following Bayesian identity (Dong and Harris, 2015):

$$p(\rho, \lambda, \boldsymbol{\beta}, \boldsymbol{\theta}, \mathbf{y}^* | \mathbf{y}) \propto \mathcal{L}(\mathbf{y}^* | \rho, \lambda, \boldsymbol{\beta}, \boldsymbol{\theta}, \sigma_u^2) \cdot \pi(\rho) \cdot \pi(\lambda) \cdot \pi(\boldsymbol{\beta}) \cdot \pi(\boldsymbol{\theta} | \lambda, \sigma_u^2) \cdot \pi(\sigma_u^2) \tag{25}$$

$$\text{posterior} \propto \text{likelihood} \times \text{prior} \tag{26}$$

where $\pi(\rho) \cdot \pi(\lambda) \cdot \pi(\boldsymbol{\beta}) \cdot \pi(\boldsymbol{\theta} | \lambda, \sigma_u^2) \cdot \pi(\sigma_u^2)$ denote the priors for the respective parameters.

The basic strategy is to employ a combination of Gibbs sampling and the Metropolis-Hasting sampling algorithms.[14] The researcher needs to be able to derive the full conditional distribution in a recognized form to be able to employ the Gibbs sampling algorithm (Smith and LeSage, 2004). As will be seen, while some sets of parameters can be estimated using the Gibbs sampling algorithm, others will have to be estimated using the Metropolis-Hastings algorithm. An additional complication in estimating ρ and λ for discrete outcomes is necessarily having to use data augmentation. Below, we reproduce the results of the derivations for the conditional distributions for the probit model by constraining σ_e^2 to 1.

Generating $\mathbf{y}^*$

We generate samples of $\mathbf{y}^*$ as follows (Albert and Chib, 1993):

$$y_{ij}^* | \rho, \boldsymbol{\beta}, \boldsymbol{\theta}, \mathbf{y} \sim \begin{cases} \mathcal{MVN}(k_i, v_i) \mathbb{1}(y_{ij}^* \geq 0) & y_{ij} = 1 \\ \mathcal{MVN}(k_i, v_i) \mathbb{1}(y_{ij}^* < 0) & y_{ij} = 0 \end{cases} \tag{27}$$

[14]Readers may refer to Jackman (2009) and Gelman et al. (2013) for a detailed reference on these algorithms.

where $\mathbb{1}$ is the indicator function, v_i denotes the $\mathbf{V}_{ii}$ element in the variance-covariance matrix of $\mathbf{y}^*$, and k_i is the i^{th} element of the $N \times 1$ column vector $\mathbf{K} \equiv \mathbf{A}^{-1}(\mathbf{X}\boldsymbol{\beta})$. Thus, equation 27 allows us to generate the latent values, y^*_{ij}, while simultaneously accounting for spatially correlated errors. Readers familiar with the data augmentation technique will note that the results above are slightly different from the case of truncated univariate normal distributions as was used for the (non-spatial) probit model by Albert and Chib (1993). The added complication in generating y^*_{ij} here in the context of spatial econometrics is the interdependence of errors (Franzese, Hays and Cook, 2016). Naively using independent truncated univariate normal distributions leads to erroneous inferences in this case. The solution to this problem is to use the truncated multivariate normal distribution (LeSage and Pace, 2009; Wilhelm and de Matos, 2013).

Conditional Posterior Distribution for $\boldsymbol{\beta}$

$$p(\boldsymbol{\beta}|\mathbf{y}^*, \rho, \lambda, \boldsymbol{\theta}, \sigma^2_u) \propto \mathcal{L}(\mathbf{y}^*|\rho, \lambda, \boldsymbol{\beta}, \boldsymbol{\theta}, \sigma^2_u) \cdot \pi(\boldsymbol{\beta}) \tag{28}$$

$$\propto \exp\left\{-\frac{1}{2}(\mathbf{A}\mathbf{y}^* - \mathbf{X}\boldsymbol{\beta} - \boldsymbol{\Delta}\boldsymbol{\theta})'(\mathbf{A}\mathbf{y}^* - \mathbf{X}\boldsymbol{\beta} - \boldsymbol{\Delta}\boldsymbol{\theta})\right\} \times \exp\left\{-\frac{1}{2}(\boldsymbol{\beta} - \mathbf{M}_0)'\mathbf{T}_0^{-1}(\boldsymbol{\beta} - \mathbf{M}_0)\right\} \tag{29}$$

$$\propto \exp\left\{-\frac{1}{2}\boldsymbol{\beta}'[\mathbf{X}'\mathbf{X} + \mathbf{T}_0^{-1}]\boldsymbol{\beta} + [(\mathbf{A}\mathbf{y}^* - \boldsymbol{\Delta}\boldsymbol{\theta})'\mathbf{X} + \mathbf{T}_0^{-1}\mathbf{M}_0]\boldsymbol{\beta} + \mathbf{C}\right\} \tag{30}$$

The basic logic behind deriving the full conditional distributions is to treat priors that do not contain the parameters of interest as constants. This allows us to simplify the original expression summarizing the relationship between the posterior on the one hand and the likelihood and the prior on the other as shown above. We can then just work with the kernel of the distributions by omitting any constants that do not affect the proportionality: equations 29 and 30 show how the conditional distribution for $\boldsymbol{\beta}$ is derived after omitting the priors for the parameters that are not of interest and working with the kernels. It is well-known that this form can be simplified further by making use of the properties of the normal distribution (Smith and LeSage, 2004). If we focus on the terms within the brackets, the expression above simplifies to the following:

$$p(\boldsymbol{\beta}|\mathbf{y}^*, \rho, \lambda, \boldsymbol{\theta}, \sigma_u^2) \sim \mathcal{MVN}(\mathbf{M}_\beta, \boldsymbol{\Sigma}_\beta) \tag{31}$$

where $\boldsymbol{\Sigma}_\beta \equiv [\mathbf{X}'\mathbf{X} + \mathbf{T}_0^{-1}]^{-1}$ and $\mathbf{M}_\beta \equiv \boldsymbol{\Sigma}_\beta[\mathbf{X}'(\mathbf{A}\mathbf{y}^* - \boldsymbol{\Delta\theta}) + \mathbf{T}_0^{-1}\mathbf{M}_0]$. Equation 31 is the simplified expression from combining the likelihood with the prior shown in equation (20). Equation 31 shows that we can use multivariate normal distribution with mean $\boldsymbol{\Sigma}_\beta$ and variance $\mathbf{M}_\beta$ to draw updated values of $\boldsymbol{\beta}$.

Conditional Posterior Distribution for $\boldsymbol{\theta}$

The logic for updating $\boldsymbol{\theta}$ is very similar to the logic for updating $\boldsymbol{\beta}$ (Dong and Harris, 2015):

$$\pi(\boldsymbol{\theta}|\lambda, \sigma_u^2) \cdot \mathcal{L}(\mathbf{y}^*|\rho, \lambda, \boldsymbol{\beta}, \boldsymbol{\theta}, \sigma_u^2) \tag{32}$$

$$\boldsymbol{\theta}|\lambda, \sigma_u^2 \sim \mathcal{MVN}(\mathbf{M}_\theta, \boldsymbol{\Sigma}_\theta) \tag{33}$$

where $\boldsymbol{\Sigma}_\theta \equiv [\boldsymbol{\Delta}'\boldsymbol{\Delta} + (\sigma_u^2)^{-1}\mathbf{B}'\mathbf{B}]^{-1}$ and $\mathbf{M}_\theta \equiv \boldsymbol{\Sigma}_\theta[\boldsymbol{\Delta}'(\mathbf{AY} - \mathbf{X}\beta)]$. Equation 32 shows the full conditional distribution for $\boldsymbol{\theta}$ from which new values may be drawn based on the updated values of λ and σ_u^2.

Conditional Posterior Distribution for $\boldsymbol{\sigma_u^2}$

The conditional posterior distribution for σ_u^2 is (Dong and Harris, 2015):

$$\sigma_u^2 \sim \mathcal{IV}\left(\frac{J}{2} + a_0, \frac{\boldsymbol{\theta}'\mathbf{B}'\mathbf{B}\boldsymbol{\theta}}{2} + b_0\right) \tag{34}$$

where $\mathcal{IV}$ denotes the inverse-gamma distribution and a_0 and b_0 are parameters set a priori by the researcher.[15]

Conditional Posterior Distribution for λ

The conditional posterior distribution for λ is (Dong and Harris, 2015):

$$p(\lambda|\mathbf{y}^*, \rho, \boldsymbol{\beta}, \sigma_u^2, \boldsymbol{\theta}) \propto \pi(\boldsymbol{\theta}|\lambda, \sigma_u^2) \cdot \pi(\lambda) \tag{35}$$

$$\propto |\mathbf{I}_J - \lambda\mathbf{M}| \times \exp\left\{-\frac{1}{2\sigma_u^2}\boldsymbol{\theta}'\mathbf{B}'\mathbf{B}\boldsymbol{\theta}\right\} \tag{36}$$

where $\mathbf{B} = \mathbf{I}_J - \lambda\mathbf{M}$. Equation 35 is not a distribution of a known form and we will have to use the Metropolis-Hastings sampling algorithm. Instead of working with the acceptance ratio directly, we use a logged-transformed version of the ratio for numerical stability purposes (Hoff, 2009).

[15] We set these to 0.01 similar to Dong and Harris (2015).

Conditional Posterior Distribution for ρ

$$p(\rho|\lambda, \boldsymbol{\beta}, \boldsymbol{\theta}, \mathbf{y}^*, \mathbf{y}) = \frac{p(\rho, \lambda, \boldsymbol{\beta}, \boldsymbol{\theta}, \mathbf{y}^*|\mathbf{y})}{p(\lambda, \boldsymbol{\beta}, \boldsymbol{\theta}, \mathbf{y}^*|\mathbf{y})} \tag{37}$$

$$\propto p(\rho, \lambda, \boldsymbol{\beta}, \boldsymbol{\theta}, \mathbf{y}^*|\mathbf{y}) \tag{38}$$

$$\propto \pi(\rho) \cdot \pi(\mathbf{y}^*|\rho, \lambda, \boldsymbol{\beta}, \boldsymbol{\theta}, \sigma_u^2) \tag{39}$$

$$\propto \det|\mathbf{I}_N - \rho\mathbf{W}| \times \exp\left\{-\frac{1}{2}\Big(\mathbf{A}\mathbf{y}^* - \mathbf{X}\boldsymbol{\beta} - \boldsymbol{\Delta}\boldsymbol{\theta})\Big)'\Big(\mathbf{A}\mathbf{y}^* - \mathbf{X}\boldsymbol{\beta} - \boldsymbol{\Delta}\boldsymbol{\theta})\Big)\right\} \tag{40}$$

where $\mathbf{A} = \mathbf{I} - \rho\mathbf{W}$. Similar to the conditional distribution of λ, equation (39) is not a distribution of a known form and we will have to use the Metropolis-Hastings sampling algorithm. Once again, we do not work with the acceptance ratio directly but instead use a logged-transformed version for the purposes of numerical stability (Hoff, 2009).

5 Model Implications

Before we delve into the Monte Carlo simulations, it is worth elaborating what our model entails. First, we note that a multilevel random intercept probit model is a special case of our model when their is no spatial interdependence at both levels (i.e., with the restrictions $\rho = 0$ and $\lambda = 0$).

$$\mathbf{y}^* = \rho\mathbf{W}\mathbf{y}^* + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\Delta}\boldsymbol{\theta} + \boldsymbol{\epsilon} \quad \boldsymbol{\theta} = (\mathbf{I} - \lambda\mathrm{M})^{-1}\mathbf{u} \tag{41}$$

$$\Rightarrow \mathbf{y}^* = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\Delta}\mathbf{u} + \boldsymbol{\epsilon} \quad \text{where} \quad \mathbf{u} \sim \mathcal{N}(\mathbf{0}, \sigma_u^2\mathbf{I}_J) \tag{42}$$

$$\tag{43}$$

The main implication here is that the multilevel random intercept probit model is likely to perform similarly to our proposed HSAR probit models when both λ and ρ are relatively low. However, we would expect the multilevel random intercept probit model to render biased estimates of $\boldsymbol{\beta}$ as ρ increases because the multilevel random intercept probit model is not properly accounting for the spatial diffusion process.

Second, when just $\lambda = 0$, note that our model simplifies into a SAR probit model with a random intercept. It is worth highlighting that this is different from the traditional SAR probit model in the spatial econometrics literature which does not contain a random intercept.

$$\mathbf{y}^* = \rho\mathbf{W}\mathbf{y}^* + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\Delta}\boldsymbol{\theta} + \boldsymbol{\epsilon} \quad \boldsymbol{\theta} = (\mathbf{I}_J - \lambda\mathbf{M})^{-1}\mathbf{u} \tag{44}$$

$$\Rightarrow \mathbf{y}^* = \rho\mathbf{W}\mathbf{y}^* + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\Delta}\mathbf{u} + \boldsymbol{\epsilon} \quad \text{where} \quad \mathbf{u} \sim \mathcal{N}(\mathbf{0}, \sigma_u^2\mathbf{I}_J) \tag{45}$$

Note that even when $\lambda = 0$, using the traditional SAR probit model will still induce bias in estimating $\boldsymbol{\beta}$ because there is a random intercept as noted above that the regular SAR probit model does not account for.[16] In this case, omitting a confounder will induce an *attenuation* bias for the independent variable of interest (Neuhaus and Jewell, 1993; Cramer, 2003). As such, although the random intercept induced by setting $\lambda = 0$ will not be correlated with

[16]At this point, it may worth reminding ourselves that the condition for inducing bias in the parameter for the independent variable of interest for binary outcomes is different from that of a linear additive model. Let us posit a linear model of the form

$$y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \epsilon_i \tag{46}$$

where X_1 is the main independent variable of interest and X_2 is the confounder. When X_2 is correlated with both X_1 and with y, we will obtain a biased estimate for β_1 if we mistakenly omit X_2. The condition of being correlated with both the independent variable variable and the dependent variable is no longer necessary to induce bias in the case of binary outcomes (Neuhaus and Jewell, 1993; Cramer, 2003).

the independent variable, not accounting for this random effect by using the traditional SAR probit model will still induce attenuation bias in estimating $\boldsymbol{\beta}$ and subsequently, an inflationary bias in estimating ρ.

If the researcher is uncertain about the existence of spatial processes, the cost of estimating our proposed HSAR solution when there are no spatial processes at both levels is efficiency losses. The benefit of treating the HSAR model as a general model is unbiasedness (assuming all other model assumptions hold) regardless of the existence of no spatial process at the both levels, a spatial process at one level, or spatial interdependence at both levels. Thus, the researcher should carefully weigh the benefits and costs, when deciding on the type of multilevel model to estimate.

6 Monte Carlo Simulations

We conduct a series of Monte Carlo simulations to assess the validity of our proposed HSAR model. We test the robustness of our model across a wide range of parameter combinations and conditions. We set the number of contiguous higher-level units such that $J = \{16, 49\}$. We generate 20 random districts within each J. This results in a combined total of $N = \{320, 980\}$ low-level units. The $N \times J$ matrix $\boldsymbol{\Delta}$ maps each of the lower level units, i, to the higher-level units, j. The $\mathbf{W}$ spatial weights matrix was generated by simulating fake counties on a map of U.S. states and using three nearest neighbors. The M spatial weights matrix was generated by using a rook contiguity matrix.

$$\mathbf{y}^* = \rho\mathbf{W}\mathbf{y}^* + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\Delta}\boldsymbol{\theta} + \boldsymbol{\epsilon} \tag{47}$$

$$\boldsymbol{\theta} = \lambda\mathbf{M}\boldsymbol{\theta} + \mathbf{u} \Rightarrow \boldsymbol{\theta} = (\mathbf{I} - \lambda\mathbf{M})^{-1}\mathbf{u} \tag{48}$$

$$\mathbf{u} \sim \mathcal{N}(\mathbf{0}, \sigma_u^2\mathbf{I}_J) \tag{49}$$

$$\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_N) \tag{50}$$

$$\boldsymbol{\theta} \sim \mathcal{N}(0, \sigma_u^2(\mathbf{B}'\mathbf{B})^{-1}) \qquad \mathbf{B} \equiv \mathbf{I}_J - \lambda\mathbf{M} \tag{51}$$

$$y_{ij} = 1 \iff y_{ij}^* \geq 0 \tag{52}$$

$$y_{ij} = 0 \iff y_{ij}^* < 0 \tag{53}$$

As discussed in detail above, the DGP may be summarized by the above equations. We vary the parameters of ρ and λ such that ρ, $\lambda \in \{0, 0.3, 0.5\}$. Thus, there are 9 unique combinations of different values for these two parameters. Note that some of these combinations render some special cases of the above DGP. For example, when both ρ and λ are equal to zero the above model becomes a random intercept probit model. When just λ equals zero, our model becomes a SAR probit model with random intercept. We set the values of the intercept (β_0) and the coefficient of X_1 (β_1) as -0.5 and 1.0, respectively similar to Wucherpfennig et al. (2021). We generate values of X_1 from an independent standard normal distribution.[17]

Since the conditional distributions of ρ and λ do not have known distributions, we adopt

[17] In Appendices B, D, F, and H, we conduct further simulations where we draw X_1 from a spatially correlated process. $X_1 = (\mathbf{I} - \rho_x\mathbf{W})^{-1}\boldsymbol{\epsilon}$ where we set ρ_x to 0.3 and $\boldsymbol{\epsilon}$ is a draw from the standard normal distribution. The results from these simulations are similar to those shown in the manuscript when X_1 is drawn from a standard normal distribution, i.e., when $\rho_x = 0$

a random walk Metropolis-Hastings for estimating these two parameters. Past research on Bayesian statistics differ on the appropriate acceptance rate for the Metropolis algorithm. We allowed the step size to auto-tune itself for the burn-in period to aim for an acceptance rate of approximately 50% per the advice of LeSage and Pace (2009). For numerical stability, we logged the acceptance ratio instead of using the ratio directly (Hoff, 2009).

The initial values for running the chain are set to 0 for all the parameters. This allows us to assess whether the algorithm allows the iteration to be sampled from values with high posterior densities. We ran 100 trials for each combination of parameters. For reasons of computational demands, we set the number of draws to 1000 for each trial and discarded the first 200 draws as a burn-in. We compare our results to the multilevel probit model implemented with the **lme4** package (Bates et al., 2015) and the SAR probit model implemented with the **ProbitSpatial** package (Martinetti et al., 2022).

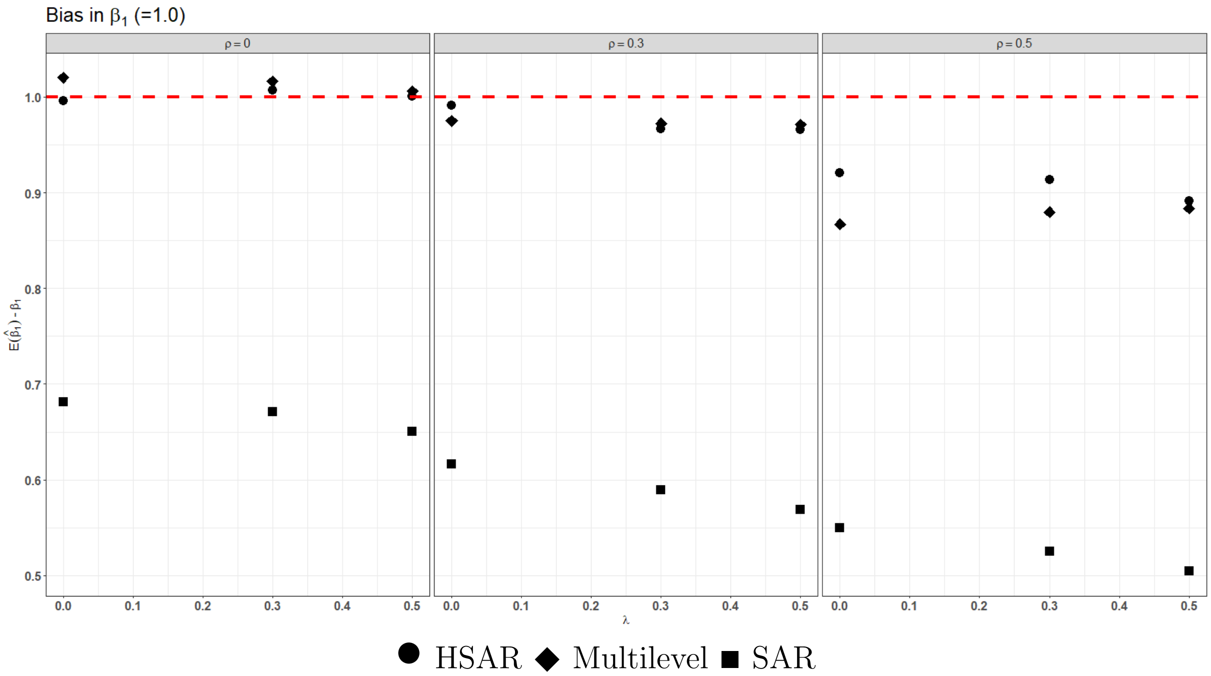
7 Monte Carlo Results

Below, we focus our discussion to the results for the binary probit case. For a given data-generating process, we estimated our proposed HSAR probit model, the traditional SAR probit model and the multilevel random intercept probit model.[18] The round circles, diamonds and squares represent the mean estimates recovered from the HSAR model, multilevel probit model and the traditional SAR model, respectively The dashed red line represents the true value of $\beta_1 = 1.0$. We first present the results for the Monte Carlo simulations with 49 high-level units and 980 low-level units with σ_u^2 set to 1 in Figure 1.

The results are consistent with our expectations. We see that there is an increasing

[18]We refer the readers to the appendix for the results for the ordered probit model.

Figure 1: Bias in $\hat{\beta}_1$ for $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$



degree of attenuation bias for the multilevel random intercept probit model as ρ increases in the true DGP because the model is incorrectly assuming that there is no spatial process. In other words, for a given level of λ, there is an increasing degree of attenuation bias for the multilevel random intercept probit model as ρ increases. We also see that the SAR probit model induces attenuation bias even when λ equals zero as expected. As discussed above, the traditional SAR probit model does not account for the random intercept and using this model incorrectly to estimate an HSAR process induces an attenuation bias for estimating β_1. We note that even using the HSAR model to estimate an HSAR process induces a noticeable degree of attenuation bias when $\rho = 0.5$. The attenuation bias that we observe here is comparable to those of past studies on spatial probit models (Franzese, Hays and Cook, 2016; Wucherpfennig et al., 2021).

The Monte Carlo results for $\hat{\rho}$ confirm our expectations as shown in Figure 2. The red dashed line represents the true values of ρ. As discussed above, the random intercept induces attenuation bias for estimating β_1, and this in turn induces an inflationary bias for estimating ρ. In particular, it is worth emphasizing that we observe this phenomenon even when $\lambda = 0$—i.e., there is no spatial process amongst the high-level units.

Figure 2: Bias in $\hat{\rho}$ for $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$

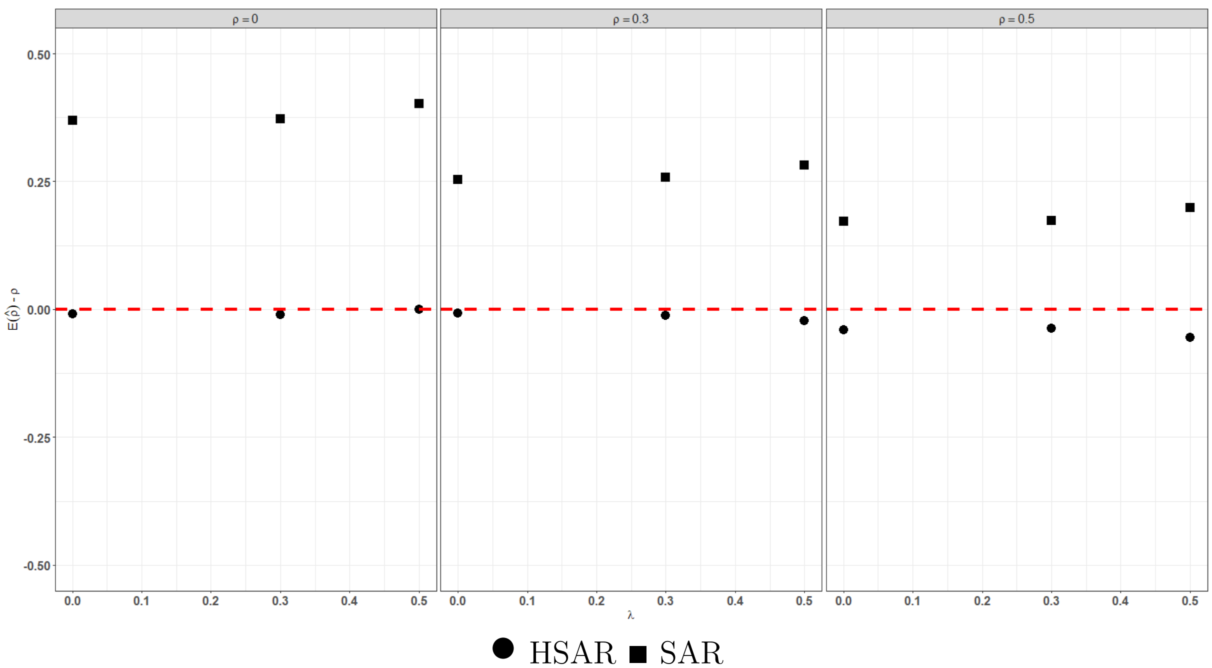
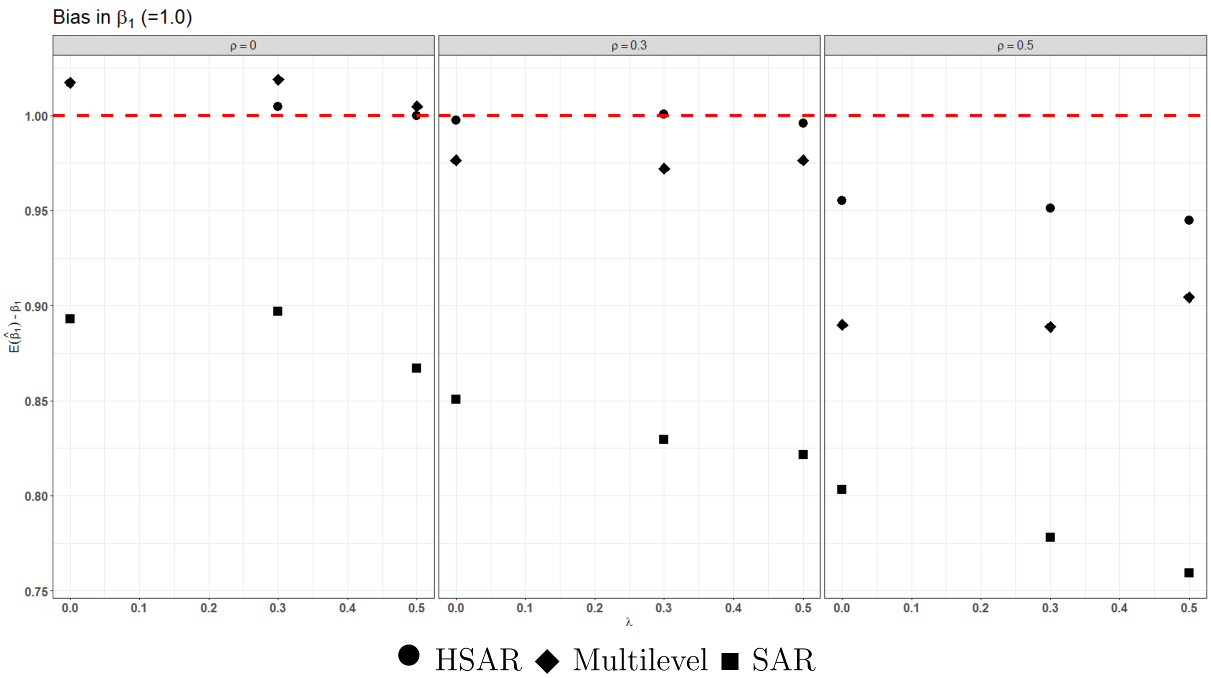


Figure 3 shows the Monte Carlo simulation results when $\sigma_u^2 = 0.5$. We see that the HSAR probit model once again shows the smallest degree of bias for estimating β_1. One difference we observe in Figure 3 is that while the multilevel random intercept probit model performed better than the traditional SAR probit model for all combinations of ρ and λ when $\sigma_u^2 = 1.0$, the multilevel model performs worse than the SAR probit model when $\rho = 0.5$ and $\sigma_u^2 = 0.5$.

We also present the results when $J = 16$, i.e. when there are 16 high-level units. Figure 4 shows that the multilevel random intercept probit model and the HSAR probit model perform similarly when $\rho = 0$. However, once again, the performance of the multilevel model becomes

Figure 3: Bias in $\hat{\beta}_1$ for J = 49, N = 980, σ_u^2 = 0.5



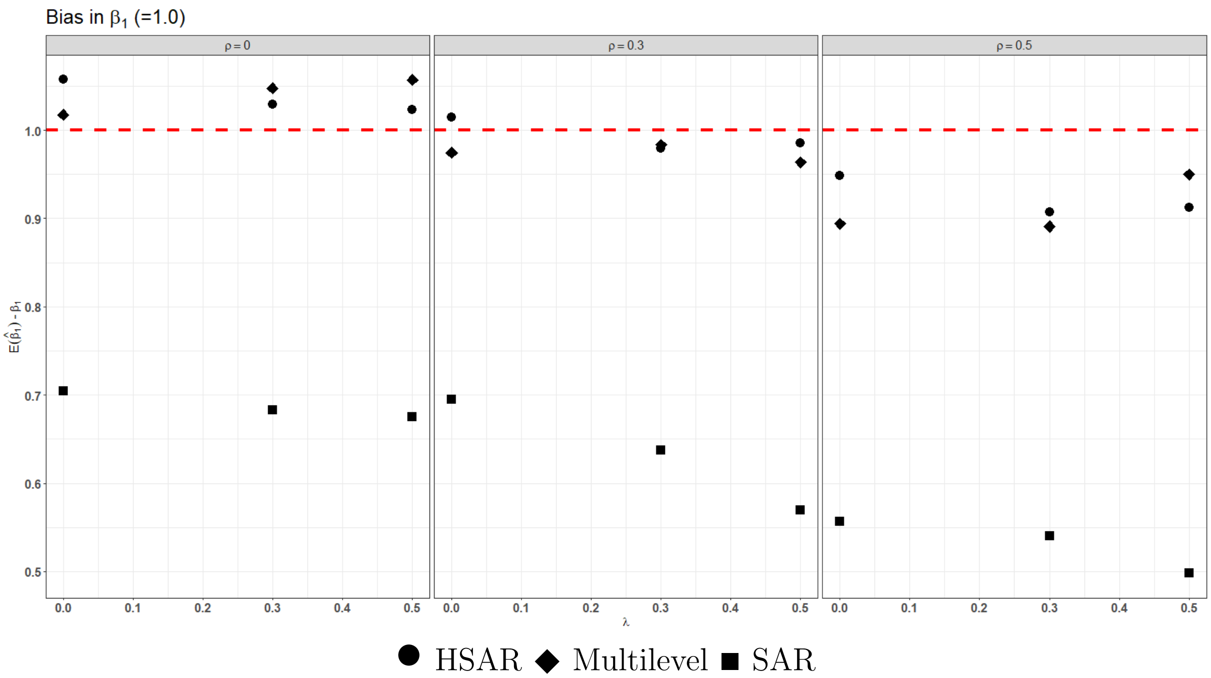
worse as ρ increases. We also see that, once again, using the traditional SAR probit model induces an attenuation bias in recovering β_1 estimates.

8 Calculation of Effects

Researchers on spatial econometrics have warned that we cannot directly infer effects from estimates (e.g., Franzese, Hays and Cook, 2016). Since the dependent variable is discrete, we would be interested in finding out, for example, how the propensity for the ith observation to experience an event would change given that there is some change in the value of a variable, x_k, for that same unit, i.[19] While the calculation of such effects are not simple, past scholars have shown the derivation for such quantities of interest (Beron and Vijverberg,

[19]It is worth highlighting that other quantities of interest can also be calculated in spatial econometrics. For example, a researcher might be interested in how the propensity for the ith observation to experience an event would change given that there is some change in the variable x_k for unit j.

Figure 4: Bias in $\hat{\beta}_1$ for $J = 16$, $N = 320$, $\sigma_u^2 = 1.0$



2004; Franzese, Hays and Cook, 2016). The derivation below is quite similar to these past derivations.[20]

In the case of the hierarchical spatial probit model, the probability of the ith observation experiencing an event is

$$p(y_i = 1|\mathbf{X}) = p([(\mathbf{I} - \rho\mathbf{W})^{-1}\mathbf{X}\boldsymbol{\beta}]_i + [(\mathbf{I} - \rho\mathbf{W})^{-1}\boldsymbol{\Delta\theta}]_i + [(\mathbf{I} - \rho\mathbf{W})^{-1}\boldsymbol{\epsilon}]_i) \tag{54}$$

$$= p([(\mathbf{I} - \rho\mathbf{W})^{-1}\mathbf{X}\boldsymbol{\beta}]_i + [(\mathbf{I} - \rho\mathbf{W})^{-1}\boldsymbol{\Delta}\mathbf{B}^{-1}\mathbf{u}]_i + [(\mathbf{I} - \rho\mathbf{W})^{-1}\boldsymbol{\epsilon}]_i) \tag{55}$$

$$= p(v_i < [(\mathbf{I} - \rho\mathbf{W})^{-1}\mathbf{X}\boldsymbol{\beta}]_i) \tag{56}$$

$$= \Phi\left\{[(\mathbf{I} - \rho\mathbf{W})^{-1}\mathbf{X}\boldsymbol{\beta}]_i / \sigma_{v_i}\right\} \tag{57}$$

where $\mathbf{B} = \mathbf{I}_J - \lambda\mathbf{M}$, Φ is the CDF of the standard normal distribution and σ_{v_i} is the iith

[20] Readers may want to compare the derivations below to those of Franzese, Hays and Cook (2016, 155,160).

element of the variance-covariance matrix $\mathbf{V} \equiv \mathbf{A}^{-1}\Big(\mathbf{\Delta}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{\Delta}' + \mathbf{I}_N\Big)(\mathbf{A}^{-1})'$ mentioned above.[21]

Let x_{ik} stand for variable x_k for observation i. To calculate the change in propensity for the ith observation to experience an event due to a change in x_{ik} (direct effect), we apply the chain rule and derive the following

$$\frac{\partial p(y_i = 1|\mathbf{X}, \mathbf{M}, \mathbf{W})}{\partial x_{ik}} = \phi_{\text{pdf}}\Big\{[(\mathbf{I} - \rho\mathbf{W})^{-1}\mathbf{X}\boldsymbol{\beta}]_i/\sigma_{v_i}\Big\}[(\mathbf{I} - \rho\mathbf{W})^{-1}\beta_k]_{ii}/\sigma_{v_i} \tag{59}$$

Similarly, the change in propensity for the ith observation to experience an event due to a change in x_k for some other unit j (indirect) would be

$$\frac{\partial p(y_i = 1|\mathbf{X}, \mathbf{M}, \mathbf{W})}{\partial x_{ik}} = \phi_{\text{pdf}}\Big\{[(\mathbf{I} - \rho\mathbf{W})^{-1}\mathbf{X}\boldsymbol{\beta}]_i/\sigma_{v_i}\Big\}[(\mathbf{I} - \rho\mathbf{W})^{-1}\beta_k]_{ij}/\sigma_{v_i} \tag{60}$$

The total effect of a change in a predictor on both its own outcome and the outcome of other units can thus be calculate by summing the direct and indirect effects.

[21]Note that

$$V[(\mathbf{I} - \rho\mathbf{W})^{-1}\mathbf{\Delta}\mathbf{B}^{-1}\mathbf{u} + (\mathbf{I} - \rho\mathbf{W})^{-1}\boldsymbol{\epsilon}] = V[(\mathbf{I} - \rho\mathbf{W})^{-1}(\mathbf{\Delta}\mathbf{B}^{-1}\mathbf{u} + \boldsymbol{\epsilon})] \tag{58}$$

renders the variance-covariance matrix $Var(\mathbf{y}^*) = \mathbf{A}^{-1}\Big(\mathbf{\Delta}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{\Delta}'+\mathbf{I}_N\Big)(\mathbf{A}^{-1})' \equiv \mathbf{V}$ we previously worked out before.

9 Extension to Ordered Outcomes

Below, we provide a summary of our extension to the ordered outcomes case. A more detailed discussion of this can be found in Appendix I. The way we conceptualize the outcome as a latent variable is exactly the same as before. These are then categorized into C different outcomes depending on the threshold cutpoints.[22]

$$\mathbf{y}^* = \rho \mathbf{W} \mathbf{y}^* + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\Delta}\boldsymbol{\theta} + \boldsymbol{\epsilon}, \tag{61}$$

We conduct a series of Monte Carlo simulations with $J = 49$ and $N = 980$. We once again find that our HSAR ordered probit performs favorably compared to the ordered spatial probit model and is comparable to the multilevel probit model in terms of recovering the estimates. We present the simulation results for the ordered outcomes in Appendix I to M due to space constraints.

10 Application

We now demonstrate the utility of our model by analyzing the diffusion process of civil rights protests in the United States in the 1960s. There are good theoretical reasons *not* to overlook the (potential) diffusion process of civil rights protests. Theoretically, scholars have debated whether and to what extent protests diffuse in various contexts (e.g., Hale, 2019). In the context of the United States' civil rights protests in the 1960s, sociologists have pointed out various mechanisms through which protests might have spread. For example, Andrews

[22] The discussion on estimating the cutpoints is relegated to Appendix I.

and Biggs (2006) argue that local newspapers played an important role in the diffusion of protests across nearby cities. At the same time, we argue that the potential diffusion process across states has to be taken into account for at least two reasons. First, cities nearby may be dispersed across different states. A cursory glance of the map showing where the sit-ins in the 1960s occurred suggests that there might have been a potential diffusion process for neighboring cities across North Carolina and Virginia. Second, historical accounts suggest that interstate diffusion process might be an important factor to take into account. For example, past research shows that activists traveled extensively with interstate buses as part of the civil rights movement (Andrews and Biggs, 2006). By no means, do we suggest that the analysis here is complete – the purpose of this exercise to compare the performance across the various models.

We use the dataset provided by Mazumder (2018) and investigate the potential causes of civil rights protests. We take various modeling approaches and compare the coefficient estimates from the different models. The unit of analysis is county in the United States. The dependent variable is whether a civil rights protest took place at least once during the period, coded as 1 if any protest took place and 0 otherwise. For our covariates, we include the percentage of urban population, the percentage of black population, the median age and the median years of school education. We use various model specifications to demonstrate the differences across the models. We first estimate the model using ordinary least squares, then sequentially estimate the model using probit, multilevel random intercept probit,[23] SAR Probit using Bayesian MCMC methods,[24] and finally our hierarchical SAR probit. The

[23]Implemented with the **lme4** package by Bates et al. (2015).

[24]Implemented with the **spatialprobit** package by Wilhelm and de Matos (2013).

random intercepts at the state level are added for the multilevel random intercept probit model and for our hierarchical SAR probit model. We try two different samples with just the 16 states in the southern part of the United States[25] and 48 states. We used 10,000 iterations and discarded the first 2,000 as a burn-in.

Table 1: Comparison of Models (16 states)

	Non-Spatial Linear	Probit	Multilevel Probit	Spatial Probit	Hierarchical Spatial Probit
Percentage of Urban Population	0.003	0.021	0.028	0.033	0.027
	(0.000)	(0.003)	(0.003)	(0.003)	(0.003)
Percentage of Black Population	0.004	0.036	0.034	0.032	0.034
	(0.000)	(0.004)	(0.005)	(0.005)	(0.005)
Median Age	-0.004	-0.018	0.000	-0.039	0.003
	(0.002)	(0.014)	(0.019)	(0.015)	(0.019)
Median School Years	0.046	0.274	0.283	0.233	0.298
	(0.008)	(0.058)	(0.075)	(0.058)	(0.072)
Constant	-0.366	-4.936	-5.737	-3.528	-5.961
	(0.078)	(0.702)	(0.947)	(0.743)	(1.093)
ρ	–	–	–	0.696	-0.093
				(0.025)	(0.083)
λ	–	–	–	–	0.690
					(0.186)
N	1378	1378	1378	1378	1378

Standard errors in parentheses

The most salient difference that stands out is the large estimates of the ρ coefficient when using the spatial autoregressive probit model as shown in Model 4 in Tables 1 and 2. On the other hand, the estimate obtained from using our hierarchical spatial probit model renders more conservative estimates. This is consistent with the simulations results that we discussed above regarding how the spatial probit model may often overestimate the ρ coefficient.

Bayesian methods offer an intuitive way to estimate the uncertainty around the effect. For each iteration after discarding the burn-in draws, we used the formula in equation (52)

[25] This includes Alabama, Arkansas, Delaware, Florida, Georgia, Kentucky, Louisiana, Maryland, Mississippi, North Carolina, Oklahoma, South Carolina, Tennessee, Texas, Virginia, and West Virginia.

Table 2: Comparison of Models (48 states)

	Non-Spatial Linear Model 1	Probit Model 2	Multilevel Probit Model 3	Spatial Probit Model 4	Hierarchical Spatial Probit Model 5
Percentage of Urban Population	0.003	0.027	0.031	0.040	0.028
	(0.000)	(0.002)	(0.002)	(0.003)	(0.002)
Percentage of Black Population	0.005	0.041	0.038	0.033	0.034
	(0.000)	(0.003)	(0.004)	(0.004)	(0.004)
Median Age	0.001	0.012	0.009	-0.023	0.010
	(0.001)	(0.010)	(0.014)	(0.010)	(0.013)
Median School Years	0.020	0.152	0.183	0.099	0.146
	(0.004)	0.042	(0.056)	(0.038)	(0.054)
Constant	-0.300	-5.015	-5.569	-3.053	-4.687
	(0.050)	(0.567)	(0.765)	(0.526)	(0.767)
ρ	–	–	–	0.700	0.152
				(0.015)	(0.072)
λ	–	–	–	–	0.452
					(0.233)
N	3043	3043	3043	3043	3043

Standard errors in parentheses

to calculate the effect. We repeated this procedure for all the observations in our data and calculated the mean over the observations to compute the average direct effect. This naturally allows us to create credible intervals for the average direct effect. For the analysis using 48 states, the average marginal effect [and the associated 95 percent simulated intervals] were found to be 0.003 [0.002, 0.004] for percentage of urban population, 0.003 [0.002, 0.004] for percentage of black population, 0.001 [-0.001, 0.004] for median age, and 0.015 [0.004, 0.027] for median school years. Noting that the quantities are those that take into account of all the spatial interaction effects, the effects obtained then can be interpreted as (direct) average marginal effect (Hanmer and Ozan Kalkan, 2013, 266). Due to space constraints, we show the distributions of the simulated effects in Appendix N.

11 Conclusion

We proposed a novel method that accounts for multilevel spatial interdependence in binary and ordered outcomes. Our proposed HSAR model is estimated using Markov Chain Monte Carlo methods and outperforms competing approaches. Overall, our Monte Carlo simulations demonstrated that in DGPs with multiple variations of spatial interdependence—no spatial interdependence in any level, spatial interdependence only in one level, or spatial interdependence in two levels—the HSAR model either performs favorably or outperforms other competing approaches. As a result, among theories and models of spatial interdependence in binary and ordered outcomes, the HSAR model can be considered a general model and could also serve as a robustness check for results from other models that capture spatial interdependence in the outcome. We also demonstrated how researchers can calculate spatial effects of substantive interest and demonstrated the utility of our model with an application to Civil Rights protest data in the United States. We provide an R package that can help researchers estimate this model and calculate the resulting substantive effects with ease.

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Online Appendix for

Modeling Hierarchical Spatial Interdependence for Limited Dependent Variables

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A Binary Probit $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$, $\rho_x = 0.0$, $\rho \in \{0, 0.3, 0.5\}$ and $\lambda \in \{0, 0.3, 0.5\}$

Table A1: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$, $\rho_x = 0.0$, $\rho = 0.0$, $\lambda = 0.0$

Experiment 1	HSAR				SAR			Multilevel	
$\rho = 0.0$, $\lambda = 0.0$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.009	-0.03	-0.005	-0.004	0.369	0.268	-0.319	-0.007	0.02
SD	0.068	0.214	0.173	0.076	0.054	0.076	0.078	0.147	0.086
RMSE	0.069	0.216	0.173	0.076	0.373	0.278	0.328	0.147	0.089

Table A2: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$, $\rho_x = 0.0$, $\rho = 0.0$, $\lambda = 0.3$

Experiment 2	HSAR				SAR			Multilevel	
$\rho = 0.0$, $\lambda = 0.3$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.011	-0.044	0.006	0.007	0.372	0.262	-0.329	-0.014	0.016
SD	0.075	0.199	0.294	0.079	0.069	0.115	0.074	0.216	0.074
RMSE	0.076	0.204	0.294	0.079	0.378	0.287	0.337	0.216	0.076

Table A3: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$, $\rho_x = 0.0$, $\rho = 0.0$, $\lambda = 0.5$

Experiment 3	HSAR				SAR			Multilevel	
$\rho = 0.0$, $\lambda = 0.5$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	0	-0.043	-0.037	0.001	0.402	0.275	-0.35	-0.013	0.006
SD	0.072	0.143	0.338	0.076	0.055	0.121	0.081	0.263	0.068
RMSE	0.072	0.149	0.34	0.076	0.405	0.3	0.359	0.264	0.069

Table A4: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$, $\rho_x = 0.0$, $\rho = 0.3$, $\lambda = 0.0$

Experiment 4	HSAR				SAR			Multilevel	
$\rho = 0.3$, $\lambda = 0.0$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.007	-0.027	0.028	-0.009	0.254	0.283	-0.384	-0.194	-0.025
SD	0.061	0.224	0.149	0.072	0.039	0.075	0.082	0.22	0.075
RMSE	0.061	0.225	0.152	0.073	0.257	0.292	0.392	0.293	0.079

Table A5: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$, $\rho_x = 0.0$, $\rho = 0.3$, $\lambda = 0.3$

Experiment 5	HSAR				SAR			Multilevel	
$\rho = 0.3$, $\lambda = 0.3$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.012	-0.057	-0.008	-0.033	0.259	0.272	-0.41	-0.217	-0.028
SD	0.055	0.19	0.222	0.08	0.039	0.09	0.076	0.27	0.084
RMSE	0.057	0.199	0.222	0.087	0.261	0.286	0.417	0.347	0.088

Table A6: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$, $\rho_x = 0.0$, $\rho = 0.3$, $\lambda = 0.5$

Experiment 6	HSAR				SAR			Multilevel	
$\rho = 0.3$, $\lambda = 0.5$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.023	-0.036	0.082	-0.034	0.282	0.279	-0.431	-0.256	-0.029
SD	0.057	0.158	0.331	0.071	0.049	0.136	0.087	0.408	0.081
RMSE	0.061	0.162	0.341	0.078	0.286	0.31	0.44	0.482	0.086

Table A7: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$, $\rho_x = 0.0$, $\rho = 0.5$, $\lambda = 0.0$

Experiment 7	HSAR				SAR			Multilevel	
$\rho = 0.5$, $\lambda = 0.0$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.04	-0.023	0.036	-0.079	0.173	0.283	-0.45	-0.303	-0.133
SD	0.05	0.203	0.156	0.084	0.036	0.072	0.102	0.245	0.081
RMSE	0.064	0.205	0.16	0.116	0.176	0.292	0.461	0.39	0.156

Table A8: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$, $\rho_x = 0.0$, $\rho = 0.5$, $\lambda = 0.3$

Experiment 8	HSAR				SAR			Multilevel	
$\rho = 0.5$, $\lambda = 0.3$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.037	-0.08	0.042	-0.086	0.174	0.276	-0.475	-0.322	-0.12
SD	0.047	0.197	0.209	0.088	0.038	0.094	0.076	0.359	0.073
RMSE	0.059	0.212	0.213	0.123	0.178	0.292	0.481	0.482	0.141

Table A9: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$, $\rho_x = 0.0$, $\rho = 0.5$, $\lambda = 0.5$

Experiment 9	HSAR				SAR			Multilevel	
$\rho = 0.5$, $\lambda = 0.5$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.055	-0.098	0.034	-0.109	0.199	0.29	-0.495	-0.403	-0.116
SD	0.049	0.167	0.276	0.094	0.042	0.129	0.104	0.516	0.092
RMSE	0.073	0.194	0.278	0.144	0.203	0.318	0.506	0.655	0.149

B Binary Probit $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$, $\rho_x = 0.3$, $\rho \in \{0, 0.3, 0.5\}$ and $\lambda \in \{0, 0.3, 0.5\}$

Table B10: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$, $\rho_x = 0.3$, $\rho = 0.0$, $\lambda = 0.0$

Experiment 1	HSAR				SAR			Multilevel	
$\rho = 0.0$, $\lambda = 0.0$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.003	-0.048	-0.015	0.011	0.348	0.258	-0.383	0.018	0.016
SD	0.07	0.229	0.161	0.077	0.056	0.085	0.075	0.164	0.076
RMSE	0.07	0.234	0.162	0.078	0.353	0.272	0.39	0.165	0.078

Table B11: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$, $\rho_x = 0.3$, $\rho = 0.0$, $\lambda = 0.3$

Experiment 2	HSAR				SAR			Multilevel	
$\rho = 0.0$, $\lambda = 0.3$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.016	-0.083	-0.029	-0.001	0.371	0.239	-0.372	-0.04	0.023
SD	0.068	0.19	0.25	0.076	0.059	0.114	0.077	0.235	0.074
RMSE	0.07	0.207	0.252	0.076	0.376	0.265	0.38	0.238	0.078

Table B12: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$, $\rho_x = 0.3$, $\rho = 0.0$, $\lambda = 0.5$

Experiment 3	HSAR				SAR			Multilevel	
$\rho = 0.0$, $\lambda = 0.5$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.011	-0.034	-0.009	0.004	0.403	0.268	-0.426	-0.033	0.013
SD	0.079	0.158	0.434	0.073	0.05	0.137	0.077	0.31	0.073
RMSE	0.079	0.162	0.434	0.073	0.406	0.301	0.433	0.312	0.074

Table B13: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$, $\rho_x = 0.3$, $\rho = 0.3$, $\lambda = 0.0$

Experiment 4	HSAR				SAR			Multilevel	
$\rho = 0.3$, $\lambda = 0.0$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.01	-0.055	-0.002	-0.016	0.25	0.239	-0.434	-0.168	0.042
SD	0.062	0.218	0.155	0.071	0.038	0.066	0.065	0.173	0.091
RMSE	0.063	0.225	0.155	0.072	0.253	0.248	0.439	0.241	0.1

Table B14: Binary Probit: J = 49, N = 980, σ_u^2 = 1.0, ρ_x = 0.3, ρ = 0.3, λ = 0.3

Experiment 5	HSAR				SAR			Multilevel	
ρ = 0.3, λ = 0.3	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.006	-0.07	0.027	-0.014	0.25	0.269	-0.448	-0.2	0.02
SD	0.049	0.2	0.203	0.077	0.044	0.107	0.083	0.306	0.083
RMSE	0.049	0.212	0.205	0.078	0.254	0.29	0.455	0.366	0.085

Table B15: Binary Probit: J = 49, N = 980, σ_u^2 = 1.0, ρ_x = 0.3, ρ = 0.3, λ = 0.5

Experiment 6	HSAR				SAR			Multilevel	
ρ = 0.3, λ = 0.5	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.017	-0.076	0.017	-0.022	0.286	0.291	-0.498	-0.169	0.038
SD	0.056	0.194	0.351	0.086	0.044	0.132	0.08	0.423	0.075
RMSE	0.058	0.208	0.352	0.089	0.289	0.32	0.505	0.456	0.084

Table B16: Binary Probit: J = 49, N = 980, σ_u^2 = 1.0, ρ_x = 0.3, ρ = 0.5, λ = 0.0

Experiment 7	HSAR				SAR			Multilevel	
ρ = 0.5, λ = 0.0	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.039	0.01	0.073	-0.095	0.15	0.249	-0.478	-0.37	-0.027
SD	0.044	0.214	0.138	0.068	0.037	0.072	0.088	0.242	0.086
RMSE	0.059	0.214	0.156	0.116	0.154	0.259	0.486	0.443	0.09

Table B17: Binary Probit: J = 49, N = 980, σ_u^2 = 1.0, ρ_x = 0.3, ρ = 0.5, λ = 0.3

Experiment 8	HSAR				SAR			Multilevel	
ρ = 0.5, λ = 0.3	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.043	-0.06	0.044	-0.095	0.18	0.282	-0.517	-0.336	-0.036
SD	0.044	0.221	0.222	0.07	0.04	0.1	0.093	0.359	0.083
RMSE	0.061	0.229	0.226	0.118	0.185	0.299	0.526	0.491	0.09

Table B18: Binary Probit: J = 49, N = 980, σ_u^2 = 1.0, ρ_x = 0.3, ρ = 0.5, λ = 0.5

Experiment 9	HSAR				SAR			Multilevel	
ρ = 0.5, λ = 0.5	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.045	-0.073	0.034	-0.104	0.19	0.289	-0.558	-0.357	-0.043
SD	0.048	0.164	0.335	0.083	0.04	0.118	0.081	0.47	0.085
RMSE	0.066	0.179	0.337	0.133	0.195	0.312	0.564	0.59	0.095

C Binary Probit $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.0$, $\rho \in \{0, 0.3, 0.5\}$ and $\lambda \in \{0, 0.3, 0.5\}$

Table C19: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.0$, $\rho = 0.0$, $\lambda = 0.0$

Experiment 1	HSAR				SAR			Multilevel	
$\rho = 0.0$, $\lambda = 0.0$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.005	-0.055	-0.011	0.017	0.155	0.122	-0.107	-0.002	0.017
SD	0.074	0.236	0.132	0.075	0.071	0.083	0.061	0.113	0.077
RMSE	0.074	0.242	0.132	0.077	0.17	0.148	0.123	0.113	0.079

Table C20: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.0$, $\rho = 0.0$, $\lambda = 0.3$

Experiment 2	HSAR				SAR			Multilevel	
$\rho = 0.0$, $\lambda = 0.3$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.02	-0.058	-0.005	0.004	0.148	0.114	-0.103	-0.009	0.019
SD	0.07	0.242	0.177	0.072	0.071	0.097	0.061	0.157	0.073
RMSE	0.073	0.249	0.177	0.072	0.164	0.15	0.12	0.157	0.076

Table C21: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.0$, $\rho = 0.0$, $\lambda = 0.5$

Experiment 3	HSAR				SAR			Multilevel	
$\rho = 0.0$, $\lambda = 0.5$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.014	-0.098	-0.058	0	0.184	0.135	-0.133	-0.006	0.005
SD	0.066	0.217	0.242	0.069	0.063	0.098	0.059	0.188	0.067
RMSE	0.068	0.238	0.249	0.069	0.195	0.167	0.146	0.188	0.067

Table C22: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.0$, $\rho = 0.3$, $\lambda = 0.0$

Experiment 4	HSAR				SAR			Multilevel	
$\rho = 0.3$, $\lambda = 0.0$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.003	-0.031	0.004	-0.003	0.119	0.138	-0.149	-0.202	-0.024
SD	0.056	0.239	0.128	0.074	0.043	0.066	0.066	0.163	0.068
RMSE	0.056	0.241	0.128	0.074	0.127	0.153	0.163	0.26	0.072

Table C23: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.0$, $\rho = 0.3$, $\lambda = 0.3$

Experiment 5	HSAR				SAR			Multilevel	
$\rho = 0.3$, $\lambda = 0.3$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.005	-0.06	-0.007	0.001	0.123	0.132	-0.17	-0.209	-0.028
SD	0.058	0.212	0.154	0.079	0.042	0.083	0.065	0.195	0.071
RMSE	0.058	0.22	0.154	0.079	0.13	0.156	0.182	0.286	0.076

Table C24: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.0$, $\rho = 0.3$, $\lambda = 0.5$

Experiment 6	HSAR				SAR			Multilevel	
$\rho = 0.3$, $\lambda = 0.5$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.006	-0.091	-0.019	-0.004	0.136	0.139	-0.179	-0.24	-0.024
SD	0.059	0.208	0.262	0.074	0.052	0.115	0.077	0.297	0.079
RMSE	0.06	0.227	0.263	0.074	0.146	0.181	0.194	0.382	0.082

Table C25: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.0$, $\rho = 0.5$, $\lambda = 0.0$

Experiment 7	HSAR				SAR			Multilevel	
$\rho = 0.5$, $\lambda = 0.0$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.016	-0.011	0.009	-0.045	0.062	0.136	-0.197	-0.332	-0.11
SD	0.048	0.212	0.118	0.076	0.032	0.064	0.088	0.186	0.079
RMSE	0.05	0.212	0.119	0.088	0.069	0.151	0.216	0.381	0.135

Table C26: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.0$, $\rho = 0.5$, $\lambda = 0.3$

Experiment 8	HSAR				SAR			Multilevel	
$\rho = 0.5$, $\lambda = 0.3$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.034	-0.052	0.006	-0.049	0.062	0.132	-0.222	-0.33	-0.111
SD	0.049	0.185	0.156	0.073	0.034	0.084	0.069	0.261	0.069
RMSE	0.059	0.192	0.156	0.087	0.07	0.156	0.232	0.421	0.131

Table C27: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.0$, $\rho = 0.5$, $\lambda = 0.5$

Experiment 9	HSAR				SAR			Multilevel	
$\rho = 0.5$, $\lambda = 0.5$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.038	-0.08	0.024	-0.055	0.081	0.148	-0.241	-0.411	-0.096
SD	0.05	0.207	0.228	0.077	0.035	0.109	0.092	0.365	0.085
RMSE	0.063	0.222	0.229	0.095	0.088	0.184	0.258	0.55	0.128

D Binary Probit $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.3$, $\rho \in \{0, 0.3, 0.5\}$ and $\lambda \in \{0, 0.3, 0.5\}$

Table D28: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.3$, $\rho = 0.0$, $\lambda = 0.0$

Experiment 1	HSAR				SAR			Multilevel	
$\rho = 0.0$, $\lambda = 0.0$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	0.013	-0.044	-0.009	0.01	0.125	0.11	-0.134	0.013	0.012
SD	0.062	0.236	0.134	0.081	0.064	0.083	0.072	0.12	0.074
RMSE	0.063	0.241	0.134	0.082	0.141	0.138	0.152	0.12	0.075

Table D29: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.3$, $\rho = 0.0$, $\lambda = 0.3$

Experiment 2	HSAR				SAR			Multilevel	
$\rho = 0.0$, $\lambda = 0.3$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.006	-0.089	-0.028	0.023	0.15	0.099	-0.127	-0.024	0.015
SD	0.059	0.204	0.179	0.071	0.07	0.101	0.063	0.174	0.07
RMSE	0.059	0.223	0.182	0.074	0.165	0.142	0.142	0.175	0.071

Table D30: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.3$, $\rho = 0.0$, $\lambda = 0.5$

Experiment 3	HSAR				SAR			Multilevel	
$\rho = 0.0$, $\lambda = 0.5$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.028	-0.091	0.001	0.013	0.174	0.12	-0.165	-0.025	0.012
SD	0.061	0.171	0.231	0.069	0.056	0.127	0.071	0.229	0.07
RMSE	0.067	0.194	0.231	0.07	0.183	0.175	0.18	0.23	0.071

Table D31: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.3$, $\rho = 0.3$, $\lambda = 0.0$

Experiment 4	HSAR				SAR			Multilevel	
$\rho = 0.3$, $\lambda = 0.0$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.005	-0.029	0.006	0.001	0.113	0.112	-0.185	-0.157	0.041
SD	0.048	0.233	0.129	0.07	0.044	0.062	0.068	0.137	0.079
RMSE	0.048	0.235	0.129	0.07	0.121	0.128	0.197	0.209	0.089

Table D32: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.3$, $\rho = 0.3$, $\lambda = 0.3$

Experiment 5	HSAR				SAR			Multilevel	
$\rho = 0.3$, $\lambda = 0.3$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.009	-0.103	-0.011	0	0.108	0.126	-0.187	-0.208	0.027
SD	0.056	0.218	0.167	0.064	0.046	0.099	0.07	0.229	0.077
RMSE	0.057	0.241	0.167	0.064	0.117	0.16	0.2	0.309	0.082

Table D33: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.3$, $\rho = 0.3$, $\lambda = 0.5$

Experiment 6	HSAR				SAR			Multilevel	
$\rho = 0.3$, $\lambda = 0.5$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.014	-0.082	0.027	-0.01	0.132	0.147	-0.229	-0.173	0.048
SD	0.053	0.207	0.237	0.07	0.042	0.113	0.074	0.296	0.072
RMSE	0.054	0.223	0.238	0.071	0.138	0.185	0.241	0.343	0.087

Table D34: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.3$, $\rho = 0.5$, $\lambda = 0.0$

Experiment 7	HSAR				SAR			Multilevel	
$\rho = 0.5$, $\lambda = 0.0$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.023	-0.024	0.019	-0.033	0.045	0.102	-0.201	-0.37	-0.014
SD	0.039	0.245	0.107	0.075	0.034	0.068	0.08	0.179	0.077
RMSE	0.046	0.247	0.108	0.082	0.056	0.123	0.217	0.411	0.078

Table D35: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.3$, $\rho = 0.5$, $\lambda = 0.3$

Experiment 8	HSAR				SAR			Multilevel	
$\rho = 0.5$, $\lambda = 0.3$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.028	-0.086	0.003	-0.05	0.06	0.133	-0.239	-0.347	-0.011
SD	0.043	0.209	0.135	0.076	0.033	0.086	0.081	0.265	0.075
RMSE	0.052	0.226	0.135	0.091	0.069	0.158	0.252	0.437	0.076

Table D36: Binary Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.3$, $\rho = 0.5$, $\lambda = 0.5$

Experiment 9	HSAR				SAR			Multilevel	
$\rho = 0.5$, $\lambda = 0.5$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.035	-0.046	0.035	-0.054	0.068	0.141	-0.272	-0.348	-0.021
SD	0.049	0.184	0.246	0.077	0.039	0.104	0.073	0.33	0.076
RMSE	0.06	0.19	0.249	0.094	0.078	0.175	0.281	0.479	0.079

E Binary Probit $J = 16$, $N = 320$, $\sigma_u^2 = 1.0$, $\rho_x = 0.0$, $\rho \in \{0, 0.3, 0.5\}$ and $\lambda \in \{0, 0.3, 0.5\}$

Table E37: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 1.0$, $\rho_x = 0.0$, $\rho = 0.0$, $\lambda = 0.0$

Experiment 1	HSAR				SAR			Multilevel	
$\rho = 0.0$, $\lambda = 0.0$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.027	-0.059	-0.024	0.058	0.345	0.208	-0.296	-0.051	0.018
SD	0.126	0.285	0.324	0.14	0.109	0.146	0.144	0.242	0.136
RMSE	0.129	0.291	0.325	0.151	0.362	0.255	0.329	0.248	0.137

Table E38: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 1.0$, $\rho_x = 0.0$, $\rho = 0.0$, $\lambda = 0.3$

Experiment 2	HSAR				SAR			Multilevel	
$\rho = 0.0$, $\lambda = 0.3$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.03	-0.159	0.004	0.03	0.365	0.245	-0.317	-0.027	0.048
SD	0.125	0.313	0.509	0.136	0.111	0.193	0.131	0.349	0.144
RMSE	0.128	0.351	0.509	0.139	0.381	0.312	0.343	0.35	0.152

Table E39: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 1.0$, $\rho_x = 0.0$, $\rho = 0.0$, $\lambda = 0.5$

Experiment 3	HSAR				SAR			Multilevel	
$\rho = 0.0$, $\lambda = 0.5$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.012	-0.161	0.007	0.023	0.384	0.277	-0.324	0.035	0.057
SD	0.12	0.264	0.732	0.137	0.132	0.276	0.137	0.546	0.122
RMSE	0.121	0.309	0.732	0.139	0.406	0.391	0.352	0.547	0.135

Table E40: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 1.0$, $\rho_x = 0.0$, $\rho = 0.3$, $\lambda = 0.0$

Experiment 4	HSAR				SAR			Multilevel	
$\rho = 0.3$, $\lambda = 0.0$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.044	-0.02	-0.09	0.015	0.25	0.245	-0.305	-0.075	-0.026
SD	0.088	0.3	0.353	0.12	0.068	0.133	0.149	0.383	0.123
RMSE	0.098	0.3	0.364	0.121	0.259	0.279	0.34	0.39	0.126

Table E41: Binary Probit: J = 16, N = 320, σ_u^2 = 1.0, ρ_x = 0.0, ρ = 0.3, λ = 0.3

Experiment 5	HSAR				SAR			Multilevel	
ρ = 0.3, λ = 0.3	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.031	-0.146	0.022	-0.02	0.265	0.281	-0.363	-0.157	-0.016
SD	0.097	0.308	0.446	0.13	0.08	0.192	0.15	0.538	0.139
RMSE	0.102	0.341	0.446	0.132	0.276	0.341	0.393	0.561	0.14

Table E42: Binary Probit: J = 16, N = 320, σ_u^2 = 1.0, ρ_x = 0.0, ρ = 0.3, λ = 0.5

Experiment 6	HSAR				SAR			Multilevel	
ρ = 0.3, λ = 0.5	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.044	-0.178	-0.018	-0.015	0.253	0.282	-0.43	-0.135	-0.036
SD	0.122	0.305	0.671	0.157	0.078	0.226	0.174	0.822	0.141
RMSE	0.13	0.353	0.671	0.158	0.264	0.361	0.464	0.833	0.146

Table E43: Binary Probit: J = 16, N = 320, σ_u^2 = 1.0, ρ_x = 0.0, ρ = 0.5, λ = 0.0

Experiment 7	HSAR				SAR			Multilevel	
ρ = 0.5, λ = 0.0	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.051	-0.003	0.001	-0.051	0.149	0.266	-0.444	-0.437	-0.106
SD	0.081	0.306	0.341	0.125	0.074	0.127	0.135	0.493	0.143
RMSE	0.096	0.306	0.341	0.135	0.167	0.294	0.464	0.659	0.178

Table E44: Binary Probit: J = 16, N = 320, σ_u^2 = 1.0, ρ_x = 0.0, ρ = 0.5, λ = 0.3

Experiment 8	HSAR				SAR			Multilevel	
ρ = 0.5, λ = 0.3	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.056	-0.134	-0.001	-0.093	0.162	0.271	-0.46	-0.343	-0.109
SD	0.086	0.315	0.465	0.141	0.075	0.181	0.15	0.7	0.137
RMSE	0.103	0.342	0.465	0.169	0.179	0.326	0.483	0.78	0.175

Table E45: Binary Probit: J = 16, N = 320, σ_u^2 = 1.0, ρ_x = 0.0, ρ = 0.5, λ = 0.5

Experiment 9	HSAR				SAR			Multilevel	
ρ = 0.5, λ = 0.5	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.085	-0.172	0.003	-0.087	0.191	0.299	-0.502	-0.234	-0.05
SD	0.109	0.273	0.71	0.155	0.074	0.236	0.177	1.337	0.156
RMSE	0.139	0.323	0.71	0.178	0.205	0.381	0.532	1.357	0.164

F Binary Probit $J = 16$, $N = 320$, $\sigma_u^2 = 1.0$, $\rho_x = 0.3$, $\rho \in \{0, 0.3, 0.5\}$ and $\lambda \in \{0, 0.3, 0.5\}$

Table F46: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 1.0$, $\rho_x = 0.3$, $\rho = 0.0$, $\lambda = 0.0$

Experiment 1	HSAR				SAR			Multilevel	
$\rho = 0.0$, $\lambda = 0.0$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.03	-0.01	-0.047	0.03	0.324	0.269	-0.34	0.025	0.011
SD	0.115	0.27	0.327	0.124	0.11	0.168	0.131	0.29	0.123
RMSE	0.118	0.27	0.33	0.128	0.342	0.317	0.365	0.291	0.123

Table F47: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 1.0$, $\rho_x = 0.3$, $\rho = 0.0$, $\lambda = 0.3$

Experiment 2	HSAR				SAR			Multilevel	
$\rho = 0.0$, $\lambda = 0.3$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.029	-0.148	-0.084	0.037	0.344	0.245	-0.37	-0.004	0.026
SD	0.134	0.3	0.521	0.126	0.098	0.199	0.136	0.38	0.129
RMSE	0.137	0.335	0.528	0.132	0.358	0.316	0.394	0.38	0.131

Table F48: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 1.0$, $\rho_x = 0.3$, $\rho = 0.0$, $\lambda = 0.5$

Experiment 3	HSAR				SAR			Multilevel	
$\rho = 0.0$, $\lambda = 0.5$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.034	-0.2	0.02	0.033	0.33	0.254	-0.355	-0.013	0.03
SD	0.114	0.259	0.64	0.153	0.117	0.251	0.137	0.478	0.141
RMSE	0.119	0.327	0.64	0.157	0.35	0.357	0.38	0.479	0.144

Table F49: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 1.0$, $\rho_x = 0.3$, $\rho = 0.3$, $\lambda = 0.0$

Experiment 4	HSAR				SAR			Multilevel	
$\rho = 0.3$, $\lambda = 0.0$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.036	-0.003	-0.038	-0.004	0.21	0.238	-0.409	-0.277	0.059
SD	0.104	0.298	0.503	0.144	0.085	0.168	0.145	0.415	0.168
RMSE	0.11	0.298	0.504	0.144	0.227	0.291	0.434	0.499	0.178

Table F50: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 1.0$, $\rho_x = 0.3$, $\rho = 0.3$, $\lambda = 0.3$

Experiment 5	HSAR				SAR			Multilevel	
$\rho = 0.3$, $\lambda = 0.3$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.049	-0.148	-0.018	-0.002	0.234	0.264	-0.418	-0.178	0.028
SD	0.115	0.313	0.624	0.134	0.089	0.211	0.129	0.477	0.144
RMSE	0.125	0.346	0.624	0.134	0.25	0.338	0.438	0.509	0.147

Table F51: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 1.0$, $\rho_x = 0.3$, $\rho = 0.3$, $\lambda = 0.5$

Experiment 6	HSAR				SAR			Multilevel	
$\rho = 0.3$, $\lambda = 0.5$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.029	-0.127	0.049	-0.009	0.254	0.258	-0.442	-0.22	0.029
SD	0.106	0.25	0.721	0.144	0.077	0.263	0.15	0.966	0.163
RMSE	0.11	0.28	0.723	0.144	0.265	0.368	0.466	0.991	0.166

Table F52: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 1.0$, $\rho_x = 0.3$, $\rho = 0.5$, $\lambda = 0.0$

Experiment 7	HSAR				SAR			Multilevel	
$\rho = 0.5$, $\lambda = 0.0$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.062	0.024	0.01	-0.065	0.164	0.251	-0.471	-0.454	-0.006
SD	0.072	0.286	0.331	0.143	0.077	0.162	0.132	0.511	0.165
RMSE	0.095	0.287	0.332	0.157	0.182	0.299	0.489	0.684	0.165

Table F53: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 1.0$, $\rho_x = 0.3$, $\rho = 0.5$, $\lambda = 0.3$

Experiment 8	HSAR				SAR			Multilevel	
$\rho = 0.5$, $\lambda = 0.3$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.059	-0.124	0.006	-0.091	0.16	0.275	-0.475	-0.247	-0.036
SD	0.084	0.293	0.438	0.127	0.067	0.178	0.208	0.677	0.152
RMSE	0.103	0.318	0.438	0.156	0.174	0.327	0.519	0.721	0.156

Table F54: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 1.0$, $\rho_x = 0.3$, $\rho = 0.5$, $\lambda = 0.5$

Experiment 9	HSAR				SAR			Multilevel	
$\rho = 0.5$, $\lambda = 0.5$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.058	-0.223	0.058	-0.082	0.172	0.244	-0.488	-0.485	0.01
SD	0.071	0.282	0.497	0.127	0.081	0.256	0.201	1.165	0.147
RMSE	0.091	0.36	0.5	0.152	0.191	0.354	0.528	1.261	0.148

G Binary Probit $J = 16$, $N = 320$, $\sigma_u^2 = 0.5$, $\rho_x = 0.0$, $\rho \in \{0, 0.3, 0.5\}$ and $\lambda \in \{0, 0.3, 0.5\}$

Table G55: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 0.5$, $\rho_x = 0.0$, $\rho = 0.0$, $\lambda = 0.0$

Experiment 1	HSAR				SAR			Multilevel	
$\rho = 0.0$, $\lambda = 0.0$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.024	-0.001	0.003	0.021	0.227	0.135	-0.174	-0.038	0.021
SD	0.12	0.298	0.285	0.116	0.122	0.141	0.133	0.177	0.131
RMSE	0.123	0.298	0.285	0.118	0.258	0.195	0.219	0.181	0.132

Table G56: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 0.5$, $\rho_x = 0.0$, $\rho = 0.0$, $\lambda = 0.3$

Experiment 2	HSAR				SAR			Multilevel	
$\rho = 0.0$, $\lambda = 0.3$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.019	-0.134	0.012	0.031	0.246	0.169	-0.194	-0.022	0.039
SD	0.124	0.299	0.35	0.133	0.125	0.183	0.126	0.25	0.136
RMSE	0.125	0.328	0.35	0.137	0.276	0.25	0.232	0.251	0.141

Table G57: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 0.5$, $\rho_x = 0.0$, $\rho = 0.0$, $\lambda = 0.5$

Experiment 3	HSAR				SAR			Multilevel	
$\rho = 0.0$, $\lambda = 0.5$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.03	-0.228	-0.01	0.044	0.269	0.2	-0.197	0.02	0.037
SD	0.125	0.303	0.539	0.136	0.144	0.258	0.12	0.383	0.12
RMSE	0.129	0.379	0.539	0.143	0.305	0.327	0.231	0.384	0.126

Table G58: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 0.5$, $\rho_x = 0.0$, $\rho = 0.3$, $\lambda = 0.0$

Experiment 4	HSAR				SAR			Multilevel	
$\rho = 0.3$, $\lambda = 0.0$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.025	-0.063	0.01	-0.005	0.178	0.171	-0.182	-0.091	-0.033
SD	0.102	0.321	0.425	0.135	0.068	0.132	0.147	0.275	0.126
RMSE	0.105	0.327	0.425	0.135	0.191	0.216	0.234	0.29	0.13

Table G59: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 0.5$, $\rho_x = 0.0$, $\rho = 0.3$, $\lambda = 0.3$

Experiment 5	HSAR				SAR			Multilevel	
$\rho = 0.3$, $\lambda = 0.3$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.025	-0.143	-0.008	0	0.196	0.213	-0.237	-0.164	-0.01
SD	0.112	0.281	0.32	0.13	0.078	0.175	0.142	0.377	0.133
RMSE	0.115	0.315	0.32	0.13	0.211	0.275	0.276	0.411	0.133

Table G60: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 0.5$, $\rho_x = 0.0$, $\rho = 0.3$, $\lambda = 0.5$

Experiment 6	HSAR				SAR			Multilevel	
$\rho = 0.3$, $\lambda = 0.5$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.026	-0.168	-0.007	0.012	0.179	0.207	-0.283	-0.125	-0.027
SD	0.094	0.275	0.416	0.144	0.075	0.215	0.168	0.511	0.138
RMSE	0.097	0.322	0.416	0.145	0.194	0.298	0.329	0.526	0.14

Table G61: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 0.5$, $\rho_x = 0.0$, $\rho = 0.5$, $\lambda = 0.0$

Experiment 7	HSAR				SAR			Multilevel	
$\rho = 0.5$, $\lambda = 0.0$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.043	-0.002	-0.034	-0.058	0.093	0.188	-0.304	-0.431	-0.102
SD	0.081	0.306	0.241	0.139	0.073	0.122	0.124	0.337	0.116
RMSE	0.092	0.306	0.244	0.151	0.118	0.224	0.328	0.547	0.154

Table G62: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 0.5$, $\rho_x = 0.0$, $\rho = 0.5$, $\lambda = 0.3$

Experiment 8	HSAR				SAR			Multilevel	
$\rho = 0.5$, $\lambda = 0.3$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.038	-0.173	0.023	-0.02	0.103	0.198	-0.323	-0.351	-0.067
SD	0.069	0.261	0.324	0.141	0.072	0.174	0.148	0.427	0.141
RMSE	0.079	0.313	0.324	0.143	0.126	0.264	0.356	0.553	0.156

Table G63: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 0.5$, $\rho_x = 0.0$, $\rho = 0.5$, $\lambda = 0.5$

Experiment 9	HSAR				SAR			Multilevel	
$\rho = 0.5$, $\lambda = 0.5$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.066	-0.236	-0.076	-0.074	0.124	0.225	-0.361	-0.215	-0.031
SD	0.097	0.287	0.495	0.143	0.069	0.226	0.17	0.703	0.134
RMSE	0.118	0.371	0.501	0.161	0.142	0.319	0.399	0.736	0.138

H Binary Probit $J = 16$, $N = 320$, $\sigma_u^2 = 0.5$, $\rho_x = 0.3$, $\rho \in \{0, 0.3, 0.5\}$ and $\lambda \in \{0, 0.3, 0.5\}$

Table H64: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 0.5$, $\rho_x = 0.3$, $\rho = 0.0$, $\lambda = 0.0$

Experiment 1	HSAR				SAR			Multilevel	
$\rho = 0.0$, $\lambda = 0.0$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.028	0.002	-0.004	0.026	0.215	0.188	-0.205	0.015	0.018
SD	0.113	0.288	0.237	0.126	0.12	0.166	0.116	0.217	0.111
RMSE	0.116	0.288	0.237	0.129	0.246	0.251	0.235	0.217	0.113

Table H65: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 0.5$, $\rho_x = 0.3$, $\rho = 0.0$, $\lambda = 0.3$

Experiment 2	HSAR				SAR			Multilevel	
$\rho = 0.0$, $\lambda = 0.3$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.011	-0.116	-0.061	0.023	0.22	0.169	-0.237	0.001	0.012
SD	0.106	0.287	0.422	0.128	0.115	0.19	0.129	0.283	0.124
RMSE	0.107	0.309	0.426	0.13	0.248	0.254	0.27	0.283	0.124

Table H66: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 0.5$, $\rho_x = 0.3$, $\rho = 0.0$, $\lambda = 0.5$

Experiment 3	HSAR				SAR			Multilevel	
$\rho = 0.0$, $\lambda = 0.5$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	0.014	-0.176	0.014	0.026	0.21	0.165	-0.203	-0.021	0.045
SD	0.097	0.252	0.47	0.134	0.125	0.242	0.146	0.346	0.138
RMSE	0.098	0.307	0.471	0.136	0.245	0.293	0.25	0.346	0.145

Table H67: Binary Probit: $J = 16$, $N = 320$, $\sigma_u^2 = 0.5$, $\rho_x = 0.3$, $\rho = 0.3$, $\lambda = 0.0$

Experiment 4	HSAR				SAR			Multilevel	
$\rho = 0.3$, $\lambda = 0.0$	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.022	-0.023	-0.02	0.011	0.132	0.158	-0.267	-0.268	0.062
SD	0.109	0.263	0.26	0.129	0.077	0.163	0.14	0.303	0.132
RMSE	0.111	0.264	0.261	0.129	0.153	0.227	0.301	0.404	0.145

Table H68: Binary Probit: J = 16, N = 320, σ_u^2 = 0.5, ρ_x = 0.3, ρ = 0.3, λ = 0.3

Experiment 5	HSAR				SAR			Multilevel	
ρ = 0.3, λ = 0.3	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.03	-0.177	-0.043	0.032	0.156	0.187	-0.277	-0.195	0.034
SD	0.092	0.308	0.36	0.139	0.091	0.2	0.128	0.352	0.137
RMSE	0.096	0.355	0.362	0.142	0.18	0.274	0.306	0.402	0.141

Table H69: Binary Probit: J = 16, N = 320, σ_u^2 = 0.5, ρ_x = 0.3, ρ = 0.3, λ = 0.5

Experiment 6	HSAR				SAR			Multilevel	
ρ = 0.3, λ = 0.5	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.016	-0.212	-0.057	0.013	0.177	0.181	-0.296	-0.182	0.012
SD	0.101	0.279	0.516	0.145	0.089	0.263	0.14	0.52	0.12
RMSE	0.102	0.35	0.519	0.146	0.198	0.319	0.328	0.551	0.12

Table H70: Binary Probit: J = 16, N = 320, σ_u^2 = 0.5, ρ_x = 0.3, ρ = 0.5, λ = 0.0

Experiment 7	HSAR				SAR			Multilevel	
ρ = 0.5, λ = 0.0	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.059	-0.045	-0.046	-0.025	0.096	0.176	-0.332	-0.452	0.021
SD	0.086	0.302	0.266	0.137	0.065	0.156	0.132	0.367	0.154
RMSE	0.105	0.305	0.27	0.139	0.116	0.235	0.357	0.582	0.156

Table H71: Binary Probit: J = 16, N = 320, σ_u^2 = 0.5, ρ_x = 0.3, ρ = 0.5, λ = 0.3

Experiment 8	HSAR				SAR			Multilevel	
ρ = 0.5, λ = 0.3	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.045	-0.15	-0.014	-0.037	0.091	0.196	-0.339	-0.252	-0.025
SD	0.068	0.283	0.314	0.133	0.068	0.171	0.197	0.452	0.145
RMSE	0.082	0.32	0.314	0.138	0.114	0.26	0.392	0.518	0.147

Table H72: Binary Probit: J = 16, N = 320, σ_u^2 = 0.5, ρ_x = 0.3, ρ = 0.5, λ = 0.5

Experiment 9	HSAR				SAR			Multilevel	
ρ = 0.5, λ = 0.5	ρ	λ	β_0	β_1	ρ	β_0	β_1	β_0	β_1
Bias	-0.063	-0.208	-0.045	-0.052	0.103	0.17	-0.348	-0.418	0.016
SD	0.095	0.313	0.492	0.144	0.071	0.241	0.192	0.808	0.123
RMSE	0.114	0.376	0.494	0.153	0.125	0.295	0.398	0.909	0.124

I Ordered Probit

Extending the algorithm applied above to ordered outcomes is relatively straightforward. In the case of regular spatial ordered probit with three categories, LeSage and Pace (2009) shows that the spatial ordered probit is a straightforward extension of the spatial binary probit model. We can apply the same algorithm as the binary probit case other than minute changes needed for generating $\mathbf{y}^*$ and estimating ϕ_c which represent the cutoff thresholds for categorizing the latent values into different discrete outcomes. Once again, if we conceptualize the outcome as a continuous latent variable, the observation y_{ij} is of category c if

$$y_{ij} = c \quad \text{if} \quad \phi_{c-1} < y_{ij}^* \leq \phi_c \tag{62}$$

LeSage and Pace (2009, 297) notes that for an ordered case of C alternatives, three values of ϕ are fixed, namely $\phi_0 = -\infty$, $\phi_1 = 0$ and $\phi_C = +\infty$ while the thresholds ϕ_c for $c = 2, ..., C - 1$ are to be estimated. The details for estimating these parameters are explained in LeSage and Pace (2009, 297-299). We use the codes from the **spatialprobit** package (Wilhelm and de Matos, 2013) for estimating these cutpoints for our hierarchical ordered spatial probit model.

We show the results for the additional simulations where $J = 49$ and $N = 980$, $\sigma_u^2 = 1.0$ similar to the binary case. We compare the results from our hierarchical ordered spatial probit model to the ordered spatial probit model implemented with the **spatialprobit** package (Wilhelm and de Matos, 2013) and the multilevel probit model implemented with the **ordi-**

nal package (Christensen, 2019).

The results for the bias in $\hat{\beta}_1$ are presented in Figure I1. We see that the hierarchical spatial ordered probit model again performs favorably compared to the ordered spatial probit model and is comparable to the multilevel probit model in terms of recovering the estimates. The results for $\hat{\rho}$ are similar to those of the binary probit case: we see that the ordered spatial probit model consistently overestimates ρ. We present other simulation results as tables below.

Figure I1: Bias in $\hat{\beta}_1$ for J = 49, N = 980, σ_u^2 = 1.0

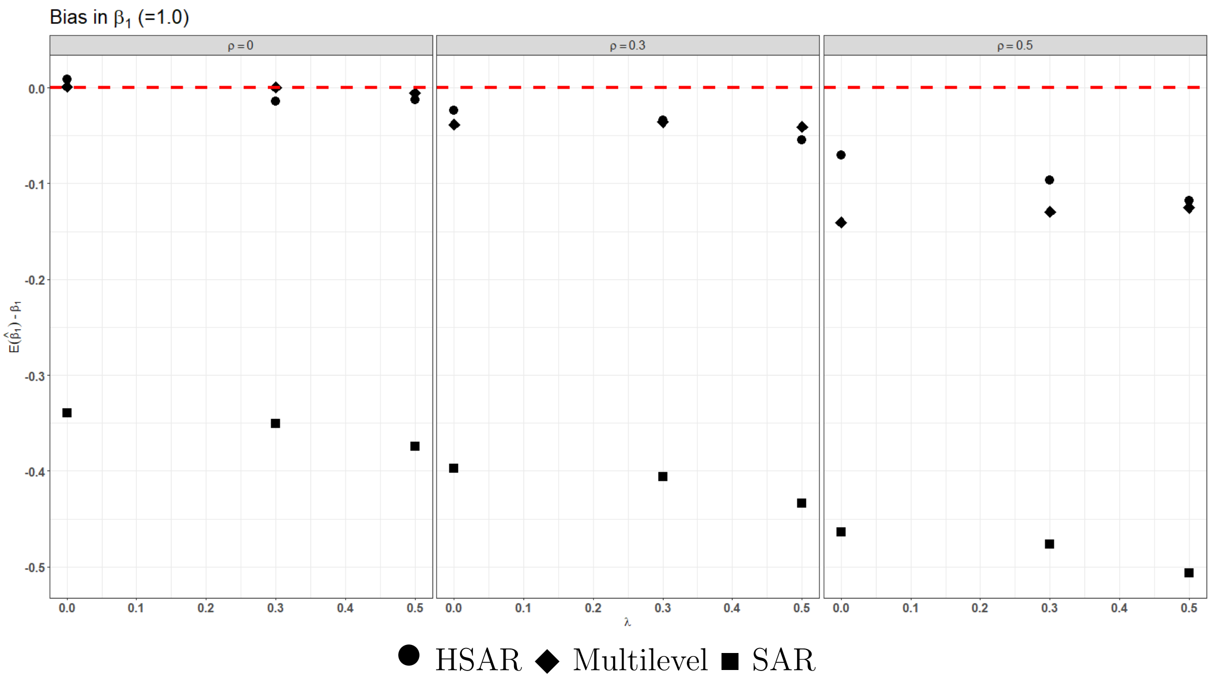
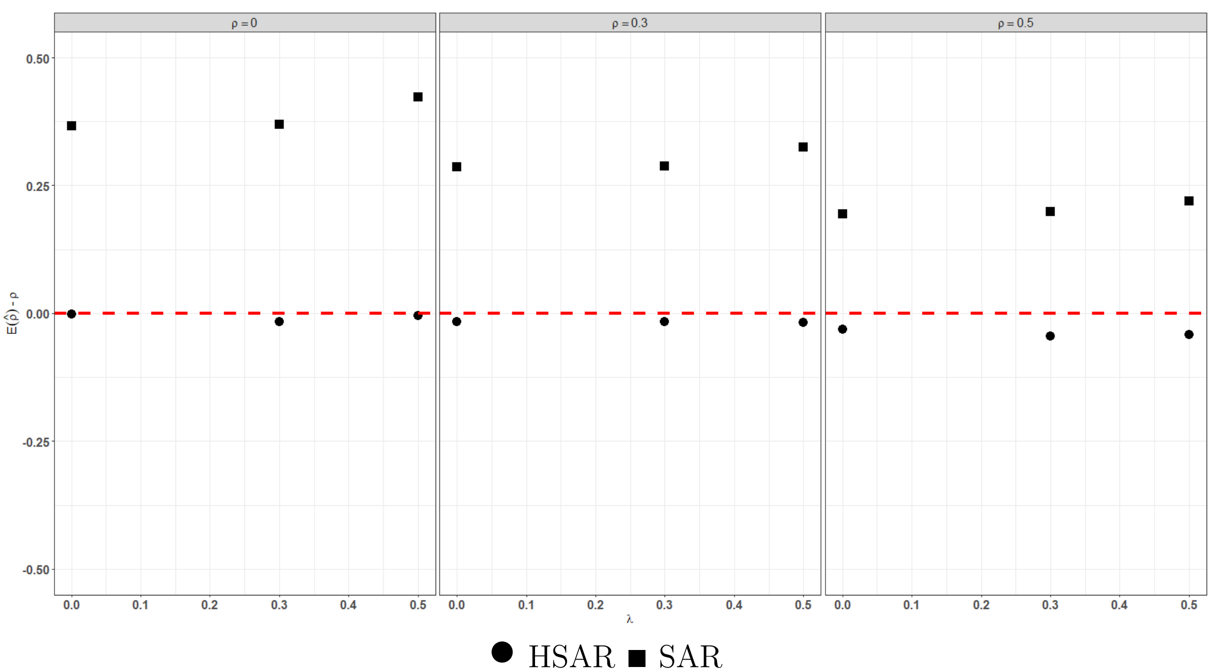


Figure I2: Bias in $\hat{\rho}$ for J = 49, N = 980, σ_u^2 = 1.0



J Ordered Probit $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$, $\rho_x = 0.0$, $\rho \in \{0, 0.3, 0.5\}$ and $\lambda \in \{0, 0.3, 0.5\}$

Table J73: Ordered Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$, $\rho_x = 0.0$, $\rho = 0.0$, $\lambda = 0.0$

Experiment 1	HSAR			SAR		Multilevel
$\rho = 0.0$, $\lambda = 0.0$	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.001	-0.038	0.009	0.367	-0.339	0.001
SD	0.058	0.222	0.075	0.065	0.047	0.071
RMSE	0.058	0.226	0.076	0.372	0.343	0.071

Table J74: Ordered Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$, $\rho_x = 0.0$, $\rho = 0.0$, $\lambda = 0.3$

Experiment 2	HSAR			SAR		Multilevel
$\rho = 0.0$, $\lambda = 0.3$	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.017	-0.065	-0.014	0.37	-0.35	0
SD	0.063	0.208	0.075	0.072	0.044	0.062
RMSE	0.065	0.218	0.077	0.377	0.353	0.062

Table J75: Ordered Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$, $\rho_x = 0.0$, $\rho = 0.0$, $\lambda = 0.5$

Experiment 3	HSAR			SAR		Multilevel
$\rho = 0.0$, $\lambda = 0.5$	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.005	-0.096	-0.013	0.423	-0.375	-0.006
SD	0.055	0.168	0.072	0.076	0.041	0.06
RMSE	0.055	0.194	0.073	0.43	0.377	0.06

Table J76: Ordered Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$, $\rho_x = 0.0$, $\rho = 0.3$, $\lambda = 0.0$

Experiment 4	HSAR			SAR		Multilevel
$\rho = 0.3$, $\lambda = 0.0$	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.017	-0.002	-0.024	0.286	-0.397	-0.039
SD	0.054	0.218	0.065	0.044	0.05	0.069
RMSE	0.057	0.218	0.069	0.289	0.4	0.079

Table J77: Ordered Probit: J = 49, N = 980, σ_u^2 = 1.0, ρ_x = 0.0, ρ = 0.3, λ = 0.3

Experiment 5	HSAR			SAR		Multilevel
ρ = 0.3, λ = 0.3	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.016	-0.075	-0.034	0.288	-0.406	-0.036
SD	0.054	0.185	0.064	0.047	0.048	0.069
RMSE	0.056	0.2	0.073	0.292	0.409	0.078

Table J78: Ordered Probit: J = 49, N = 980, σ_u^2 = 1.0, ρ_x = 0.0, ρ = 0.3, λ = 0.5

Experiment 6	HSAR			SAR		Multilevel
ρ = 0.3, λ = 0.3	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.017	-0.042	-0.054	0.325	-0.433	-0.041
SD	0.048	0.179	0.078	0.05	0.05	0.069
RMSE	0.051	0.184	0.095	0.329	0.436	0.08

Table J79: Ordered Probit: J = 49, N = 980, σ_u^2 = 1.0, ρ_x = 0.0, ρ = 0.5, λ = 0.0

Experiment 7	HSAR			SAR		Multilevel
ρ = 0.5, λ = 0.0	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.031	0.007	-0.071	0.195	-0.464	-0.141
SD	0.034	0.21	0.067	0.029	0.052	0.072
RMSE	0.046	0.21	0.097	0.197	0.467	0.158

Table J80: Ordered Probit: J = 49, N = 980, σ_u^2 = 1.0, ρ_x = 0.0, ρ = 0.5, λ = 0.3

Experiment 8	HSAR			SAR		Multilevel
ρ = 0.5, λ = 0.3	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.045	-0.049	-0.097	0.199	-0.476	-0.13
SD	0.04	0.18	0.068	0.03	0.051	0.066
RMSE	0.06	0.186	0.118	0.201	0.479	0.145

Table J81: Ordered Probit: J = 49, N = 980, σ_u^2 = 1.0, ρ_x = 0.0, ρ = 0.5, λ = 0.5

Experiment 9	HSAR			SAR		Multilevel
ρ = 0.5, λ = 0.5	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.042	-0.095	-0.118	0.219	-0.507	-0.125
SD	0.042	0.203	0.072	0.032	0.058	0.073
RMSE	0.06	0.224	0.138	0.221	0.51	0.145

K Ordered Probit $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$, $\rho_x = 0.3$, $\rho \in \{0, 0.3, 0.5\}$ and $\lambda \in \{0, 0.3, 0.5\}$

Table K82: Ordered Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$, $\rho_x = 0.3$, $\rho = 0.0$, $\lambda = 0.0$

Experiment 1	HSAR			SAR		Multilevel
$\rho = 0.0$, $\lambda = 0.0$	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.001	-0.034	-0.013	0.343	-0.404	0.003
SD	0.054	0.209	0.067	0.064	0.049	0.07
RMSE	0.054	0.212	0.068	0.349	0.407	0.07

Table K83: Ordered Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$, $\rho_x = 0.3$, $\rho = 0.0$, $\lambda = 0.3$

Experiment 2	HSAR			SAR		Multilevel
$\rho = 0.0$, $\lambda = 0.3$	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.002	-0.059	-0.011	0.367	-0.416	0.012
SD	0.056	0.2	0.068	0.065	0.05	0.064
RMSE	0.057	0.208	0.069	0.373	0.419	0.065

Table K84: Ordered Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$, $\rho_x = 0.3$, $\rho = 0.0$, $\lambda = 0.5$

Experiment 3	HSAR			SAR		Multilevel
$\rho = 0.0$, $\lambda = 0.5$	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.016	-0.056	-0.029	0.396	-0.427	0.002
SD	0.063	0.161	0.06	0.071	0.052	0.066
RMSE	0.065	0.171	0.067	0.402	0.43	0.066

Table K85: Ordered Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 1.0$, $\rho_x = 0.3$, $\rho = 0.3$, $\lambda = 0.0$

Experiment 4	HSAR			SAR		Multilevel
$\rho = 0.3$, $\lambda = 0.0$	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.012	-0.015	-0.027	0.266	-0.455	0.028
SD	0.047	0.218	0.061	0.042	0.048	0.072
RMSE	0.048	0.218	0.066	0.269	0.458	0.077

Table K86: Ordered Probit: J = 49, N = 980, σ_u^2 = 1.0, ρ_x = 0.3, ρ = 0.3, λ = 0.3

Experiment 5	HSAR			SAR		Multilevel
ρ = 0.3, λ = 0.3	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.013	-0.016	-0.035	0.278	-0.464	0.015
SD	0.044	0.167	0.063	0.046	0.051	0.071
RMSE	0.045	0.168	0.072	0.281	0.467	0.073

Table K87: Ordered Probit: J = 49, N = 980, σ_u^2 = 1.0, ρ_x = 0.3, ρ = 0.3, λ = 0.5

Experiment 6	HSAR			SAR		Multilevel
ρ = 0.3, λ = 0.5	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.025	-0.08	-0.059	0.303	-0.481	0.021
SD	0.049	0.175	0.073	0.048	0.053	0.067
RMSE	0.055	0.193	0.094	0.307	0.484	0.071

Table K88: Ordered Probit: J = 49, N = 980, σ_u^2 = 1.0, ρ_x = 0.3, ρ = 0.5, λ = 0.0

Experiment 7	HSAR			SAR		Multilevel
ρ = 0.5, λ = 0.0	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.033	-0.022	-0.086	0.177	-0.513	-0.04
SD	0.033	0.213	0.067	0.028	0.045	0.07
RMSE	0.046	0.214	0.109	0.179	0.515	0.08

Table K89: Ordered Probit: J = 49, N = 980, σ_u^2 = 1.0, ρ_x = 0.3, ρ = 0.5, λ = 0.3

Experiment 8	HSAR			SAR		Multilevel
ρ = 0.5, λ = 0.3	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.034	0.004	-0.09	0.187	-0.525	-0.042
SD	0.038	0.21	0.066	0.03	0.051	0.074
RMSE	0.051	0.21	0.111	0.189	0.528	0.085

Table K90: Ordered Probit: J = 49, N = 980, σ_u^2 = 1.0, ρ_x = 0.3, ρ = 0.5, λ = 0.5

Experiment 9	HSAR			SAR		Multilevel
ρ = 0.5, λ = 0.5	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.037	-0.106	-0.105	0.203	-0.544	-0.053
SD	0.034	0.185	0.075	0.032	0.054	0.071
RMSE	0.05	0.213	0.129	0.205	0.547	0.089

L Ordered Probit $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.0$, $\rho \in \{0, 0.3, 0.5\}$ and $\lambda \in \{0, 0.3, 0.5\}$

Table L91: Ordered Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.0$, $\rho = 0.0$, $\lambda = 0.0$

Experiment 1	HSAR			SAR		Multilevel
$\rho = 0.0$, $\lambda = 0.0$	ρ	λ	β_1	ρ	β_1	β_1
Bias	0.004	-0.073	-0.002	0.126	-0.221	0.004
SD	0.057	0.229	0.061	0.056	0.043	0.067
RMSE	0.057	0.241	0.061	0.138	0.226	0.068

Table L92: Ordered Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.0$, $\rho = 0.0$, $\lambda = 0.3$

Experiment 2	HSAR			SAR		Multilevel
$\rho = 0.0$, $\lambda = 0.3$	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.01	-0.045	-0.001	0.123	-0.226	0.004
SD	0.059	0.23	0.065	0.066	0.04	0.063
RMSE	0.06	0.234	0.065	0.139	0.229	0.063

Table L93: Ordered Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.0$, $\rho = 0.0$, $\lambda = 0.5$

Experiment 3	HSAR			SAR		Multilevel
$\rho = 0.0$, $\lambda = 0.5$	ρ	λ	β_1	ρ	β_1	β_1
Bias	0.008	-0.077	0.006	0.162	-0.252	-0.002
SD	0.068	0.181	0.071	0.066	0.039	0.058
RMSE	0.069	0.197	0.071	0.175	0.255	0.058

Table L94: Ordered Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.0$, $\rho = 0.3$, $\lambda = 0.0$

Experiment 4	HSAR			SAR		Multilevel
$\rho = 0.3$, $\lambda = 0.0$	ρ	λ	β_1	ρ	β_1	β_1
Bias	0	-0.039	0.009	0.107	-0.245	-0.034
SD	0.051	0.259	0.068	0.049	0.049	0.066
RMSE	0.051	0.262	0.069	0.117	0.25	0.074

Table L95: Ordered Probit: J = 49, N = 980, σ_u^2 = 0.5, ρ_x = 0.0, ρ = 0.3, λ = 0.3

Experiment 5	HSAR			SAR		Multilevel
ρ = 0.3, λ = 0.3	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.005	-0.077	0.001	0.097	-0.252	-0.038
SD	0.044	0.206	0.066	0.053	0.042	0.06
RMSE	0.044	0.22	0.066	0.111	0.256	0.071

Table L96: Ordered Probit: J = 49, N = 980, σ_u^2 = 0.5, ρ_x = 0.0, ρ = 0.3, λ = 0.5

Experiment 6	HSAR			SAR		Multilevel
ρ = 0.3, λ = 0.5	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.011	-0.076	-0.007	0.129	-0.275	-0.031
SD	0.055	0.177	0.074	0.055	0.042	0.073
RMSE	0.056	0.193	0.074	0.141	0.279	0.079

Table L97: Ordered Probit: J = 49, N = 980, σ_u^2 = 0.5, ρ_x = 0.0, ρ = 0.5, λ = 0.0

Experiment 7	HSAR			SAR		Multilevel
ρ = 0.5, λ = 0.0	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.023	0.011	-0.031	0.078	-0.288	-0.116
SD	0.037	0.224	0.067	0.035	0.052	0.067
RMSE	0.043	0.224	0.073	0.086	0.293	0.134

Table L98: Ordered Probit: J = 49, N = 980, σ_u^2 = 0.5, ρ_x = 0.0, ρ = 0.5, λ = 0.3

Experiment 8	HSAR			SAR		Multilevel
ρ = 0.5, λ = 0.3	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.027	-0.081	-0.043	0.068	-0.298	-0.113
SD	0.047	0.202	0.068	0.039	0.045	0.063
RMSE	0.054	0.217	0.08	0.079	0.301	0.13

Table L99: Ordered Probit: J = 49, N = 980, σ_u^2 = 0.5, ρ_x = 0.0, ρ = 0.5, λ = 0.5

Experiment 9	HSAR			SAR		Multilevel
ρ = 0.5, λ = 0.5	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.021	-0.076	-0.058	0.095	-0.322	-0.104
SD	0.038	0.216	0.065	0.043	0.052	0.069
RMSE	0.044	0.229	0.087	0.105	0.327	0.125

M Ordered Probit $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.3$, $\rho \in \{0, 0.3, 0.5\}$ and $\lambda \in \{0, 0.3, 0.5\}$

Table M100: Ordered Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.3$, $\rho = 0.0$, $\lambda = 0.0$

Experiment 1	HSAR			SAR		Multilevel
$\rho = 0.0$, $\lambda = 0.0$	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.004	-0.045	0	0.108	-0.255	0.001
SD	0.058	0.242	0.067	0.054	0.047	0.07
RMSE	0.058	0.247	0.067	0.121	0.26	0.07

Table M101: Ordered Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.3$, $\rho = 0.0$, $\lambda = 0.3$

Experiment 2	HSAR			SAR		Multilevel
$\rho = 0.0$, $\lambda = 0.3$	ρ	λ	β_1	ρ	β_1	β_1
Bias	0.005	-0.083	0.001	0.131	-0.261	0.009
SD	0.059	0.206	0.06	0.06	0.042	0.061
RMSE	0.059	0.223	0.06	0.144	0.264	0.061

Table M102: Ordered Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.3$, $\rho = 0.0$, $\lambda = 0.5$

Experiment 3	HSAR			SAR		Multilevel
$\rho = 0.0$, $\lambda = 0.5$	ρ	λ	β_1	ρ	β_1	β_1
Bias	0.003	-0.125	-0.009	0.135	-0.271	0.004
SD	0.061	0.197	0.073	0.059	0.045	0.065
RMSE	0.061	0.234	0.073	0.147	0.275	0.065

Table M103: Ordered Probit: $J = 49$, $N = 980$, $\sigma_u^2 = 0.5$, $\rho_x = 0.3$, $\rho = 0.3$, $\lambda = 0.0$

Experiment 4	HSAR			SAR		Multilevel
$\rho = 0.3$, $\lambda = 0.0$	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.018	-0.068	-0.006	0.087	-0.286	0.029
SD	0.041	0.225	0.064	0.043	0.044	0.066
RMSE	0.045	0.235	0.064	0.097	0.29	0.072

Table M104: Ordered Probit: J = 49, N = 980, σ_u^2 = 0.5, ρ_x = 0.3, ρ = 0.3, λ = 0.3

Experiment 5	HSAR			SAR		Multilevel
ρ = 0.3, λ = 0.3	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.02	-0.073	-0.002	0.1	-0.293	0.02
SD	0.039	0.21	0.069	0.044	0.046	0.069
RMSE	0.044	0.222	0.069	0.109	0.297	0.072

Table M105: Ordered Probit: J = 49, N = 980, σ_u^2 = 0.5, ρ_x = 0.3, ρ = 0.3, λ = 0.5

Experiment 6	HSAR			SAR		Multilevel
ρ = 0.3, λ = 0.5	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.015	-0.057	-0.005	0.11	-0.309	0.032
SD	0.045	0.185	0.067	0.051	0.046	0.064
RMSE	0.048	0.193	0.067	0.121	0.313	0.071

Table M106: Ordered Probit: J = 49, N = 980, σ_u^2 = 0.5, ρ_x = 0.3, ρ = 0.5, λ = 0.0

Experiment 7	HSAR			SAR		Multilevel
ρ = 0.5, λ = 0.0	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.013	-0.061	-0.044	0.054	-0.331	-0.022
SD	0.037	0.206	0.068	0.034	0.045	0.064
RMSE	0.039	0.215	0.081	0.064	0.334	0.068

Table M107: Ordered Probit: J = 49, N = 980, σ_u^2 = 0.5, ρ_x = 0.3, ρ = 0.5, λ = 0.3

Experiment 8	HSAR			SAR		Multilevel
ρ = 0.5, λ = 0.3	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.029	-0.06	-0.041	0.067	-0.345	-0.023
SD	0.039	0.183	0.07	0.032	0.046	0.067
RMSE	0.049	0.193	0.081	0.074	0.348	0.071

Table M108: Ordered Probit: J = 49, N = 980, σ_u^2 = 0.5, ρ_x = 0.3, ρ = 0.5, λ = 0.5

Experiment 9	HSAR			SAR		Multilevel
ρ = 0.5, λ = 0.5	ρ	λ	β_1	ρ	β_1	β_1
Bias	-0.028	-0.093	-0.061	0.077	-0.356	-0.029
SD	0.034	0.171	0.064	0.039	0.054	0.069
RMSE	0.044	0.195	0.089	0.087	0.36	0.075

N Application

Figure N3: Distribution of the Simulated Effect of Urban Population (16 states)

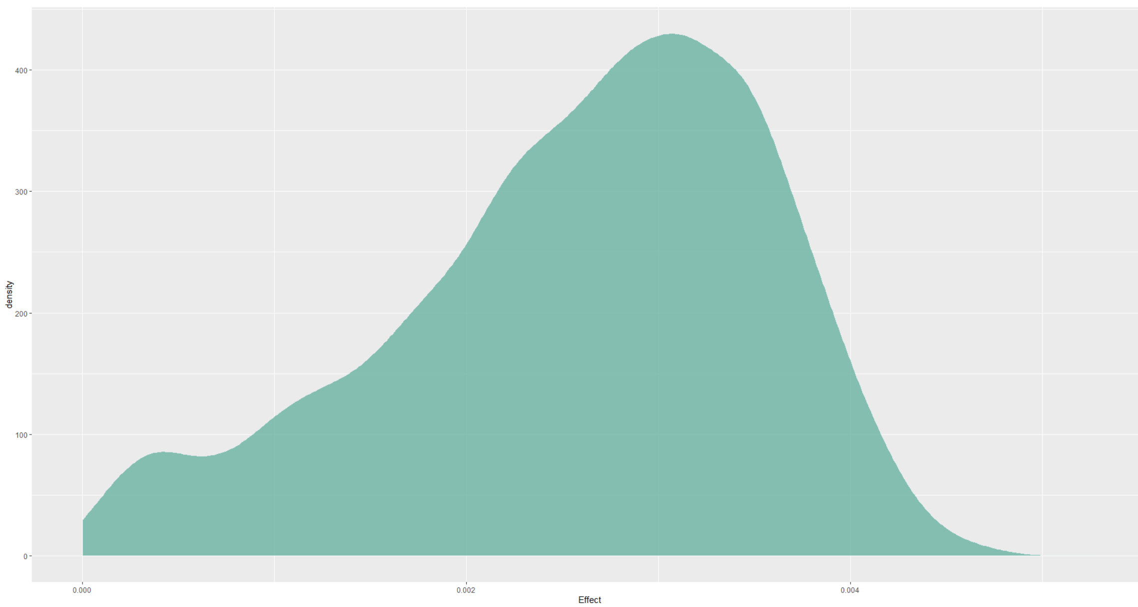


Figure N4: Distribution of the Simulated Effect of Black Population (16 states)

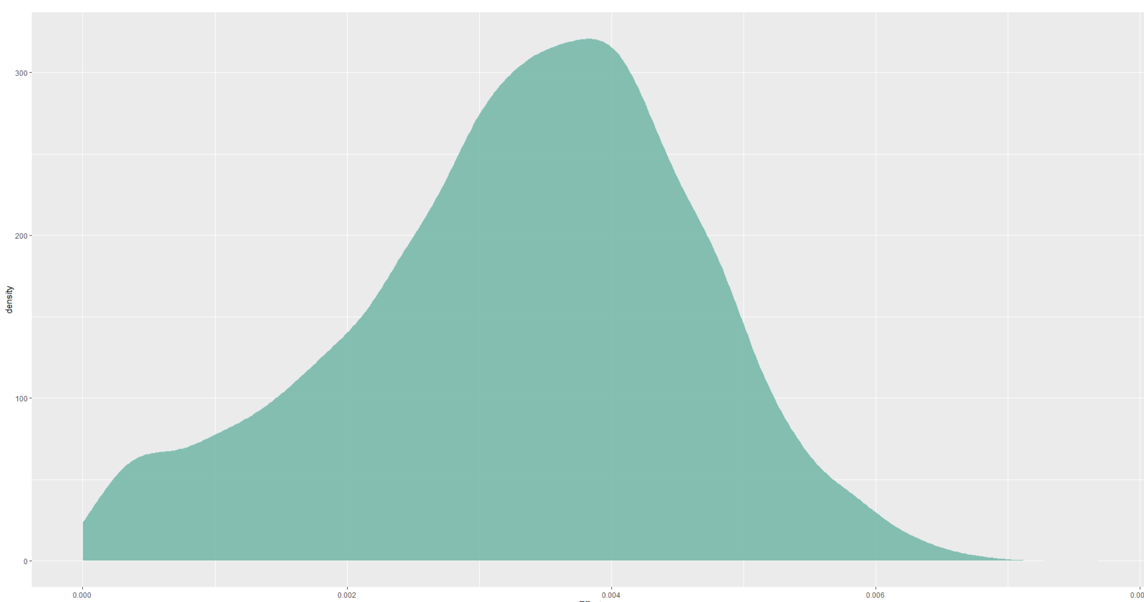


Figure N5: Distribution of the Simulated Effect of Median Age (16 states)

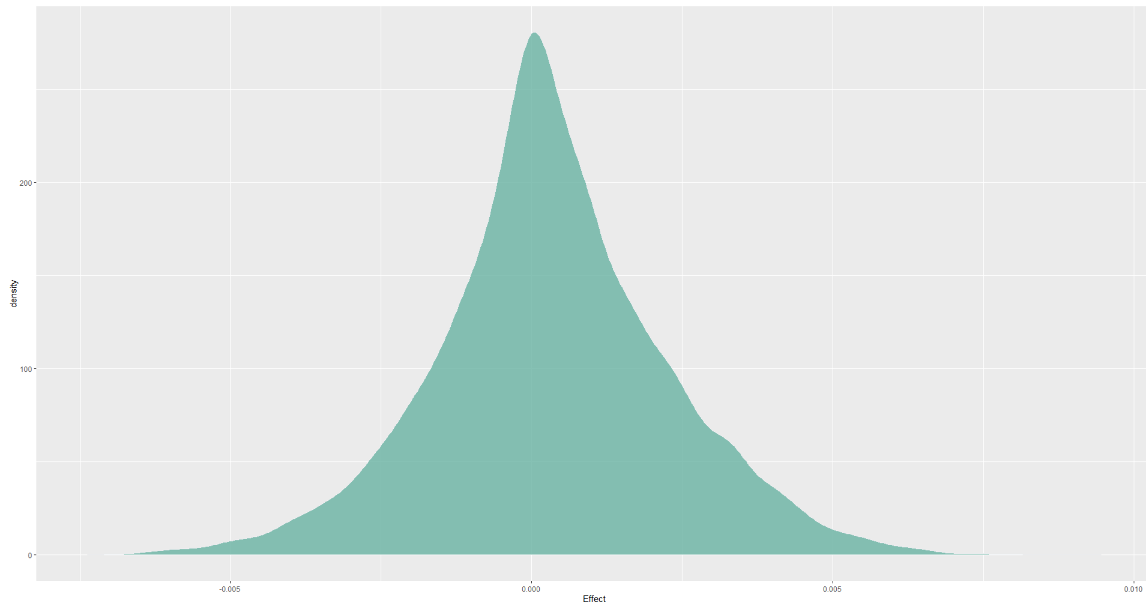


Figure N6: Distribution of the Simulated Effect of Median Years of School Education (16 states)

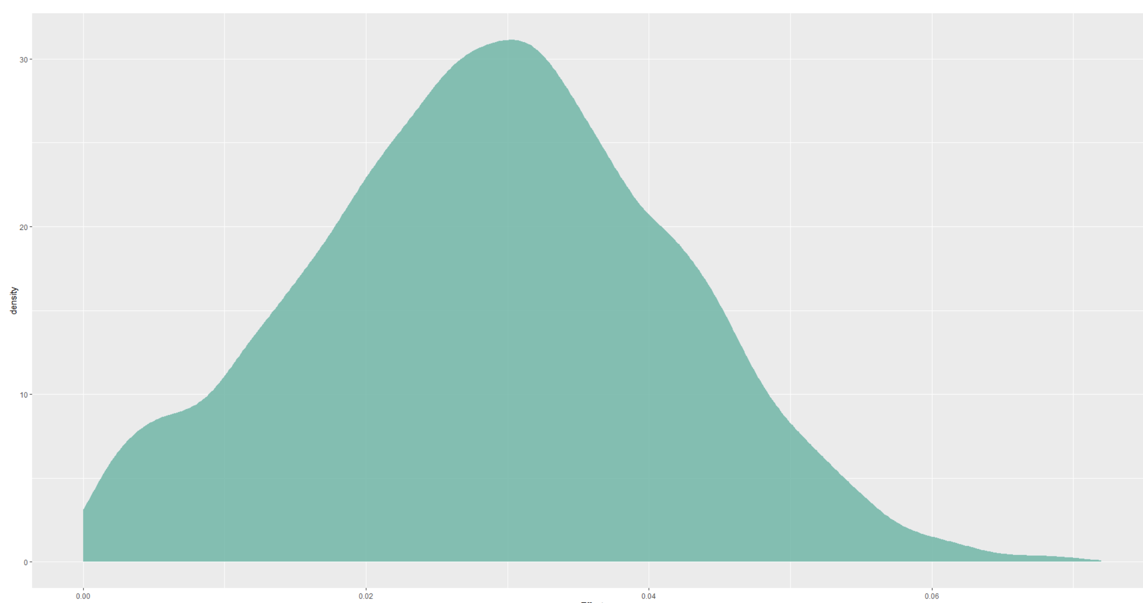


Figure N7: Distribution of the Simulated Effect of Urban Population (48 states)

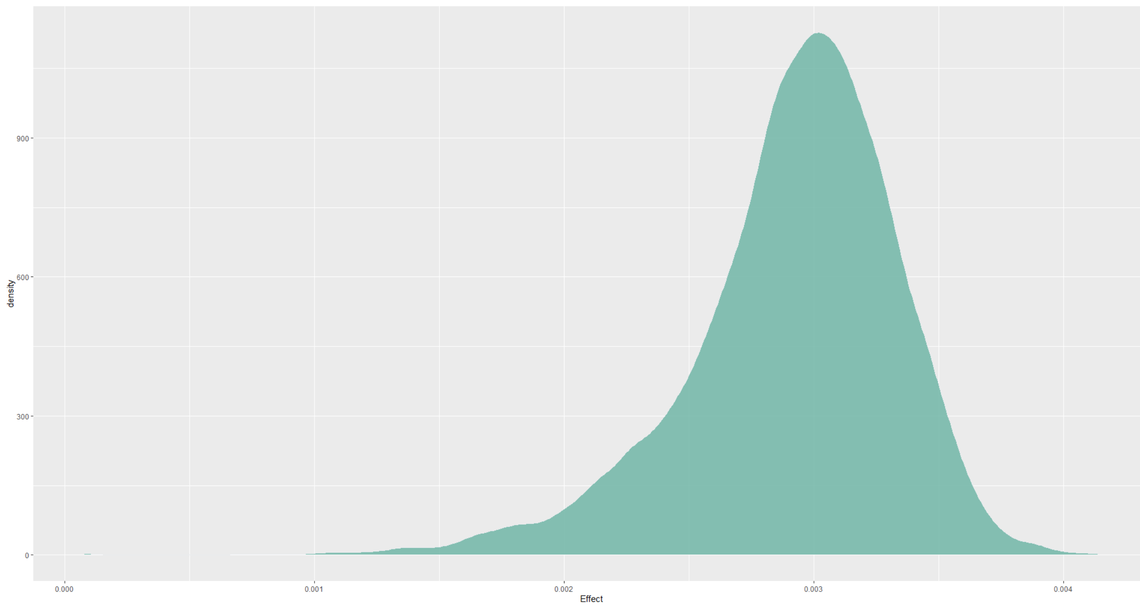


Figure N8: Distribution of the Simulated Effect of Black Population (48 states)

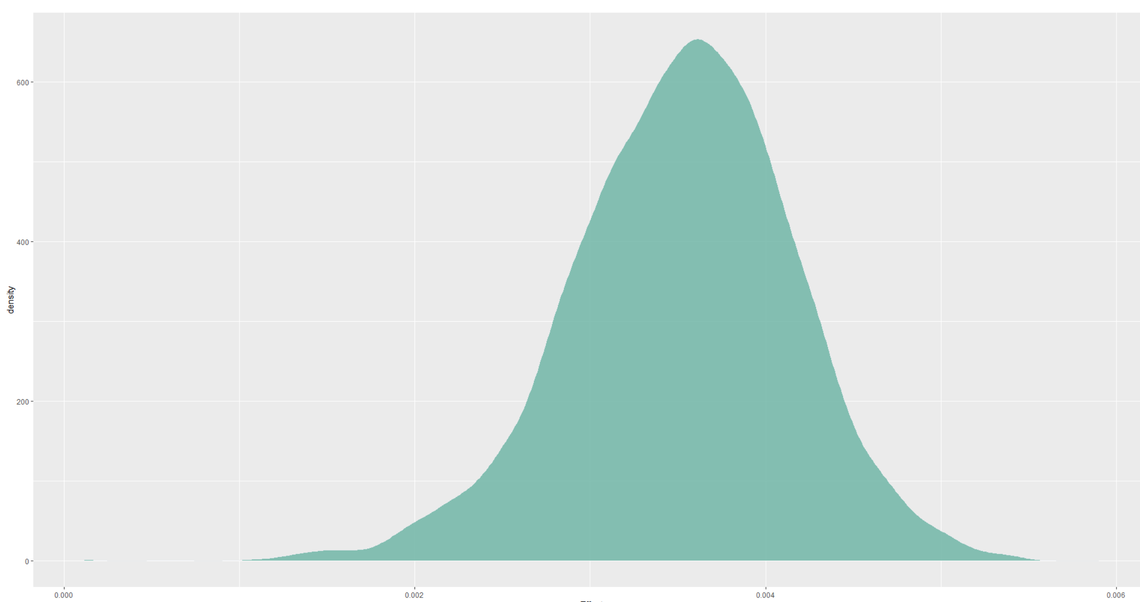


Figure N9: Distribution of the Simulated Effect of Median Age (48 states)

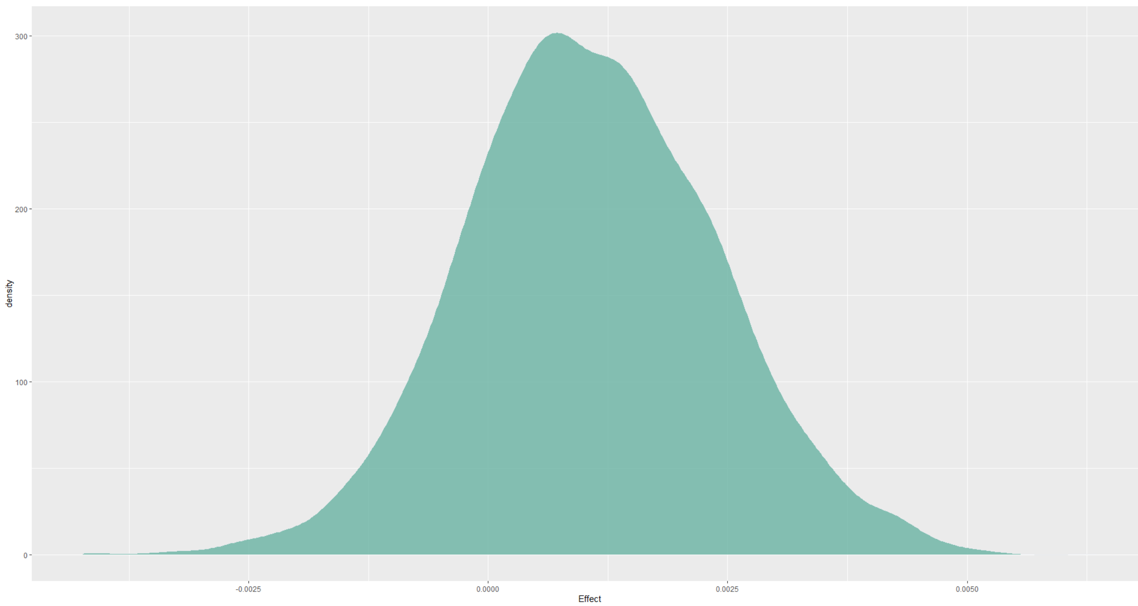


Figure N10: Distribution of the Simulated Effect of Median Years of School Education (48 states)